

# Gauging spacetime symmetries on the worldsheet and the geometric Langlands program — II

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**ABSTRACT:** We generalise the analysis carried out in [1], and find that our previous results can be extended beyond the case of  $SL(N, \mathbf{C})$ . In particular, we show that an equivalence — at the level of the holomorphic chiral algebra — between a bosonic string on a smooth coset manifold  $G/B$  and a  $B$ -gauged version of itself on  $G$ , will imply an isomorphism of classical  $\mathcal{W}$ -algebras and a level relation which underlie a geometric Langlands correspondence for the simply-laced, complex  $ADE$ -groups. In addition, as opposed to line operators and branes of an open topological sigma-model, the Hecke operators and Hecke eigensheaves, can, instead, be physically interpreted in terms of the correlation functions of local operators in the holomorphic chiral algebra of a *closed, quasi-topological* sigma-model. Our present results thus serve as an *alternative* physical interpretation — to that of an electric-magnetic duality of four-dimensional gauge theory demonstrated earlier by Kapustin and Witten in [2]—of the geometric Langlands correspondence for complex  $ADE$ -groups. The cases with tame and mild “ramifications” are also discussed.

**KEYWORDS:** BRST Quantization, Differential and Algebraic Geometry, Conformal and W Symmetry, Topological Field Theories.

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**1. Introduction**

The geometric Langlands correspondence has recently been given an elegant physical interpretation by Kapustin and Witten in their seminal paper [2]—by considering a certain twisted  $\mathcal{N} = 4$  supersymmetric Yang-Mills theory in four-dimensions compactified on a complex Riemann surface  $C$ , the geometric Langlands correspondence associated to a holomorphic  $G$ -bundle on  $C$  can be shown to arise naturally from an electric-magnetic duality in four-dimensions. Specifically, it was first argued that one can, among other things,

relate various mathematical objects and concepts of the correspondence such as Hecke eigensheaves and the action of the Hecke operator, to the boundaries and the 't Hooft line operator of the underlying four-dimensional quantum gauge theory. It was then shown that the map between the various ingredients which defines the mathematical correspondence, is nothing but a four-dimensional electric-magnetic duality, or equivalently, a mirror symmetry of the resulting two-dimensional topological sigma-model at low-energies. The framework outlined in [2] thus furnishes a purely physical interpretation of the geometric Langlands conjecture.

The work of Kapustin and Witten centres around a gauge-theoretic interpretation of the geometric Langlands correspondence. However, it does not shed any light on the utility of two-dimensional *axiomatic* conformal field theory in the geometric Langlands program, which, incidentally, is ubiquitous in the mathematical literature on the subject [3–8]. This seems rather puzzling. Afterall, the various axiomatic definitions of a conformal field theory that fill the mathematical literature, are based on established physical concepts, and it is therefore natural to expect that in any physical interpretation of the geometric Langlands correspondence, a two-dimensional conformal field theory of some sort will be involved. It will certainly be illuminating for the geometric Langlands program as a whole, if one can deduce the axiomatic conformal field-theoretic approach developed in the mathematical literature, from the gauge-theoretic approach of Kapustin and Witten, or vice-versa.

In the axiomatic conformal field-theoretic approach to the geometric Langlands correspondence, the key ingredients are affine Lie algebras at the critical level without stress tensors [9], and  $\mathcal{W}$ -algebras (defined by a Drinfeld-Sokolov or DS reduction procedure) associated to the affine versions of the Langlands dual of the Lie algebras [9, 10]. The duality between *classical*  $\mathcal{W}$ -algebras — which underlies the axiomatic conformal field-theoretic approach to the correspondence — is just an isomorphism between the *Poisson* algebra generated by the centre  $\mathfrak{z}(\widehat{\mathfrak{g}})$  of the *completed universal enveloping algebra* of the affine Lie algebra  $\widehat{\mathfrak{g}}$  at the *critical* level, where  $\mathfrak{g}$  is the Lie algebra of the group  $G$ , and the classical  $\mathcal{W}$ -algebra associated to the affine Lie algebra  ${}^L\widehat{\mathfrak{g}}$  in the limit of large level  $k'$  —  $\mathcal{W}_\infty({}^L\widehat{\mathfrak{g}})$ , where  ${}^L\mathfrak{g}$  is the Lie algebra of the Langlands dual group  ${}^LG$ ; in other words, a geometric Langlands correspondence for  $G$  simply originates from an isomorphism  $\mathfrak{z}(\widehat{\mathfrak{g}}) \simeq \mathcal{W}_\infty({}^L\widehat{\mathfrak{g}})$  of *Poisson* algebras [8, 11]. This statement is accompanied by a relation  $(k + h^\vee)r^\vee = (k' + {}^Lh^\vee)^{-1}$  between the generic levels  $k$  and  $k'$  of  $\widehat{\mathfrak{g}}$  and  ${}^L\widehat{\mathfrak{g}}$  respectively (where  $r^\vee$  is the lacing number of  $\mathfrak{g}$ , and  $h^\vee$  and  ${}^Lh^\vee$  are the dual Coxeter numbers of  $\mathfrak{g}$  and  ${}^L\mathfrak{g}$ ).

Note that the gauge-theoretic approach to the program necessarily involves a certain two-dimensional quantum field theory in its formulation, a generalised topological sigma-model to be exact. This strongly suggests that perhaps a good starting point towards elucidating the connection between the axiomatic conformal field-theoretic and gauge-theoretic approaches, would be to explore other physical models which are *purely* two-dimensional, that will enable us to make direct contact with the central results of the correspondence derived from the axiomatic conformal field-theoretic approach.

A strong hint that one should be considering for this purpose a two-dimensional twisted  $(0, 2)$  sigma-model on a flag manifold, stems from our recent understanding of the role sheaves of “Chiral Differential Operators” (or CDO’s) play in the description of its holo-

morphic chiral algebra [12], and from the fact that global sections of CDO's on a flag manifold furnish a module of an affine Lie algebra at the critical level [12, 13]. On the other hand, since Toda field theories lead to free-field realisations of the  $\mathcal{W}$ -algebras defined by the DS reduction scheme mentioned above (see section 6 of [17], and the references therein), and since the Toda theory can be obtained as a gauge-invariant content of a certain gauged WZW theory [19, 20], it should be true that a physical manifestation of the isomorphism of (classical)  $\mathcal{W}$ -algebras which underlie the geometric Langlands correspondence, ought to be given by some relation between the sigma-model on a flag manifold and a gauged WZW model. This was the main motivation for the work in [1], which represents a modest attempt towards an analysis of the relation between quantum field theory and the geometric Langlands correspondence from a purely two-dimensional viewpoint, wherein a twisted  $(0, 2)$  sigma-model on a complex flag manifold of  $SL(N, \mathbf{C})$  was considered.

In this paper, we shall generalise the analysis in [1], and show that our previous results can be extended beyond  $SL(N, \mathbf{C})$  to include all complex simply-laced groups. In particular, we shall show that an equivalence — at the level of the *holomorphic* chiral algebra — between a bosonic string on a smooth coset manifold  $G/B$  and a  $B$ -gauged version of itself on  $G$ , will necessarily imply an isomorphism  $\mathfrak{z}(\widehat{\mathfrak{g}}) \simeq \mathcal{W}_\infty({}^L\widehat{\mathfrak{g}})$  of classical  $\mathcal{W}$ -algebras and the relation  $(k+h^\vee)r^\vee = (k'+{}^Lh^\vee)^{-1}$  which underlie a geometric Langlands correspondence for  $G$ , where  $G$  is any simply-laced, complex  $ADE$ -group. This equivalence in the spectra of the bosonic strings — which can be viewed as a consequence of the ubiquitous notion that one can always physically interpret a geometrical symmetry of the target space as a gauge symmetry in the worldsheet theory — thus furnishes an *alternative* physical interpretation, to that of an electric-magnetic duality of four-dimensional gauge theory, of the geometric Langlands correspondence for the complex  $ADE$ -groups! In addition, as in [1], the Hecke operators and Hecke eigensheaves of the geometric Langlands program for  $G$ , can also be shown to lend themselves to different physical interpretations altogether — instead of line operators and branes in a two-dimensional topological sigma-model, they are, in our context, related to the correlation functions of local operators that span the holomorphic chiral algebra of a *closed* and *quasi-topological* sigma-model in two-dimensions. Moreover, the cases with tame and mild “ramifications” can also be understood from a purely physical perspective via these local operators. Our results therefore open up an alternative way of looking at the correspondence from a purely two-dimensional quantum field-theoretic standpoint, which could potentially lead to novel mathematical and physical insights for the geometric Langlands program as a whole.

**A brief summary and plan of the paper.** We shall now give a brief summary and plan of the paper.

In section 2, we begin by considering the twisted  $(0, 2)$  sigma-model on a complex flag manifold given by the coset space  $G/B$ , where  $G$  is any simply-laced, complex  $ADE$ -group with  $\mathfrak{g} = {}^L\mathfrak{g}$ , and  $B$  is a Borel subgroup containing upper triangular matrices of  $G$ . We will show that the Casimir fields spanning the *classical* holomorphic chiral algebra of the purely bosonic sector of the sigma-model, will have Laurent modes that generate the *classical* centre  $\mathfrak{z}(\widehat{\mathfrak{g}})$  of the completed universal enveloping algebra of the affine  $G$ -algebra at the critical level.

In section 3, we discuss the *dual* description of the holomorphic chiral algebra of the purely bosonic sector of the sigma-model on  $G/B$ , given by the *holomorphic* BRST-cohomology (or chiral algebra) of a  $B$ -gauged WZW model on  $G$ . We then show that the holomorphic BRST-cohomology of the  $B$ -gauged WZW model on  $G$  at level  $k'$  physically realises, in all generality, the Hecke algebra of local operators — generated by a mathematical Drinfeld-Sokolov reduction procedure [9]—which defines  $\mathcal{W}_{k'}(\widehat{\mathfrak{g}})$ , the  $\mathcal{W}$ -algebra associated to  $\widehat{\mathfrak{g}}$  at level  $k'$ .

In section 4, we use the results in the earlier sections to show that an equivalence — at the level of the holomorphic chiral algebra — between a bosonic string on  $G/B$  and a  $B$ -gauged version of itself on  $G$ , will necessarily imply an isomorphism  $\mathfrak{z}(\widehat{\mathfrak{g}}) \simeq \mathcal{W}_\infty({}^L\widehat{\mathfrak{g}})$  of classical  $\mathcal{W}$ -algebras and the relation  $(k+h^\vee)r^\vee = (k'+{}^Lh^\vee)^{-1}$  which underlie a geometric Langlands correspondence for  $G$ .

In section 5, we shall derive, via the isomorphism  $\mathfrak{z}(\widehat{\mathfrak{g}}) \simeq \mathcal{W}_\infty({}^L\widehat{\mathfrak{g}})$  of classical  $\mathcal{W}$ -algebras, a correspondence between flat holomorphic  ${}^L G$ -bundles on the worldsheet  $\Sigma$  and Hecke eigensheaves on the moduli space  $\text{Bun}_G$  of holomorphic  $G$ -bundles on  $\Sigma$ . Then, we shall physically interpret the Hecke eigensheaves and Hecke operators of the geometric Langlands program in terms of the correlation functions of purely bosonic local operators in the holomorphic chiral algebra of the twisted  $(0, 2)$  sigma-model on the complex flag manifold  $G/B$ .

In section 6, we shall briefly discuss the physical interpretation of the geometric Langlands correspondence for  $G$  with tame and mild “ramifications”, in our setting.

**Relation to the gauge-theoretic approach.** Though we have not made any explicit connections to the gauge-theoretic approach of Kapustin and Witten yet, we hope to be able to address this important issue in a later publication, perhaps with the insights gained in this paper.

## 2. The twisted $(0, 2)$ sigma-model on $G/B$ and the classical centre $\mathfrak{z}(\widehat{\mathfrak{g}})$

In this section, we consider the twisted  $(0, 2)$  sigma-model on a complex flag manifold given by the coset space  $G/B$ , where  $G$  is any complex  $ADE$ -group and  $B$  is a Borel subgroup containing upper triangular matrices of  $G$ . Via a mathematical theorem in [13], and the interpretation of the  $\overline{\mathcal{Q}}_+$ -cohomology of the sigma-model as the Cech-cohomology of the sheaf of CDO’s (as reviewed in appendix A of [1]), we explain why the scaling dimension-one operators in the holomorphic chiral algebra of the purely bosonic sector of the sigma-model will generate an affine  $G$  OPE-algebra at the critical level. We then explain why the Casimir fields constructed out of these dimension-one currents must span the *classical* holomorphic chiral algebra of the purely bosonic sector of the sigma-model, which, in turn, implies that their Laurent modes must generate the *classical* centre  $\mathfrak{z}(\widehat{\mathfrak{g}})$  of the completed universal enveloping algebra of the affine  $G$ -algebra at the critical level.

### 2.1 The twisted sigma-model on $G/B$ and the sheaf of CDO’s

As reviewed in appendix A of [1], the  $\overline{\mathcal{Q}}_+$ -cohomology or the holomorphic chiral algebra of the twisted  $(0, 2)$  sigma-model on  $X$  can be expressed in terms of the Cech-cohomology of

the sheaf of CDO's. Since our main discussion involves the holomorphic chiral algebra of the sigma-model, and since we shall need to turn to some mathematical theorems regarding the sheaf of CDO's in our arguments, we shall first describe the sigma-model in terms of the sheaf of CDO's.

Recall that  $X = G/B$ , where  $B$  is the subgroup of upper triangular matrices of  $G$  with a nilpotent Lie algebra  $\mathfrak{b}$ . Let us cover  $X$  with  $N$  open charts  $U_w$  where  $w = 1, 2, \dots, N$ , such that each open chart  $U_w$  can be identified with the affine space  $\mathbb{C}^n$ , where  $n = \dim_{\mathbb{C}} X$ . Then, as explained in appendix A of [1], the sheaf of CDO's in any  $U_w$ —which describes a localised version of the sigma-model on  $U_w$ —can be described by  $n$  free  $\beta\gamma$  systems with the action

$$I = \sum_{i=1}^n \frac{1}{2\pi} \int |d^2 z| \beta_i \partial_{\bar{z}} \gamma^i. \tag{2.1}$$

As before, the  $\beta_i$ 's and  $\gamma^i$ 's are fields of dimension  $(1,0)$  and  $(0,0)$  respectively. They obey the standard free-field OPE's; there are no singularities in the operator products  $\beta_i(z) \cdot \beta_i(z')$  and  $\gamma^i(z) \cdot \gamma^i(z')$ , while

$$\beta_i(z) \gamma^j(z') \sim -\frac{\delta_i^j}{z - z'}. \tag{2.2}$$

Similarly, the sheaf of CDO's in a neighbouring intersecting chart  $U_{w+1}$  is described by  $n$  free  $\tilde{\beta}\tilde{\gamma}$  systems with action

$$I = \sum_{i=1}^n \frac{1}{2\pi} \int |d^2 z| \tilde{\beta}_i \partial_{\bar{z}} \tilde{\gamma}^i, \tag{2.3}$$

where the  $\tilde{\beta}_i$  and  $\tilde{\gamma}^i$  fields obey the same OPE's as the  $\beta_i$  and  $\gamma^i$  fields. In other words, the non-trivial OPE's are given by

$$\tilde{\beta}_i(z) \tilde{\gamma}^j(z') \sim -\frac{\delta_i^j}{z - z'}. \tag{2.4}$$

In order to describe a globally-defined sheaf of CDO's, one will need to glue the free conformal field theories with actions (2.1) and (2.3) in the overlap region  $U_w \cap U_{w+1}$  for every  $w = 1, 2, \dots, N$ , where  $U_{1+N} = U_1$ . To do so, one must use the admissible automorphisms of the free conformal field theories defined in (A.29)-(A.30) of [1] to glue the free-fields together; they are given by

$$\tilde{\gamma}^i = [g]^i_j \gamma^j, \tag{2.5}$$

$$\tilde{\beta}_i = \beta_k D^k_i + \partial_z \gamma^j E_{ij}, \tag{2.6}$$

where  $i, j, k = 1, 2, \dots, n$ . Here,  $g$ ,  $D$  and  $E$  are  $n \times n$  matrices, whereby  $[g]$  is the matrix of geometrical transition functions,  $[(D^T)^{-1}]_i^k = \partial_i [g]^k_j \gamma^j$  and  $[E]_{ij} = \partial_i B_j$ . It can be verified that  $\tilde{\beta}$  and  $\tilde{\gamma}$  will obey the correct OPE's among themselves [13]. Moreover, let  $R_w$  represent a transformation of the fields in going from  $U_w$  to  $U_{w+1}$ . One can also verify that there is no anomaly to a global definition of a sheaf of CDO's on  $X = G/B$ —a careful computation

will reveal that one will get the desired composition maps  $(R_N \dots R_4 R_3 R_2 R_1) \cdot \gamma^j = \gamma^j$  and  $(R_N \dots R_4 R_3 R_2 R_1) \cdot \beta_i = \beta_i$ . Again, this is just a statement that one can always define a sheaf  $\widehat{\mathcal{O}}_X^{\text{ch}}$  of CDO's on any flag manifold  $X = G/B$  [13]. Physically, this just corresponds to the fact that since  $p_1(X) = 0$ , the sigma-model will be well-defined and anomaly-free (see appendix A of [1]).

## 2.2 Global sections of $\widehat{\mathcal{O}}_X^{\text{ch}}$ and an affine $G$ -algebra at the critical level

Since  $X = G/B$  is of complex dimension  $n$ , the chiral algebra  $\mathcal{A}$  of the sigma-model will be given by  $\mathcal{A} = \bigoplus_{g_R=0}^{g_R=n} H^{g_R}(X, \widehat{\mathcal{O}}_X^{\text{ch}})$  as a vector space. As in [1], it would suffice for our purpose to concentrate on the fermion-independent sector of  $\mathcal{A}$ —from our  $\overline{Q}_+$ -Cech cohomology dictionary (explained in appendix A of [1]), this again translates to studying only the global sections in  $H^0(X, \widehat{\mathcal{O}}_X^{\text{ch}})$ .

According to theorem 5.13 of [13], one can always find elements in  $H^0(X, \widehat{\mathcal{O}}_X^{\text{ch}})$  for any flag manifold  $X = G/B$ , that will furnish a module of an affine  $G$ -algebra at the critical level. This means that one can always find dimension-one global sections of the sheaf  $\widehat{\mathcal{O}}_X^{\text{ch}}$  that correspond to  $\psi^{\tilde{i}}$ -independent currents  $J^a(z)$  for  $a = 1, 2, \dots, \dim(\mathfrak{g})$ , that satisfy the OPE's of an affine  $G$ -algebra at the critical level  $k = -h^\vee$ :

$$J_a(z)J_b(z') \sim -\frac{h^\vee d_{ab}}{(z-z')^2} + \sum_c f_{ab}^c \frac{J_c(z')}{(z-z')}, \tag{2.7}$$

where  $h^\vee$  is the dual Coxeter number of the Lie algebra  $\mathfrak{g}$ , and  $d_{ab}$  is its Cartan-Killing metric.<sup>1</sup> Since these current operators correspond to global sections, it will be true that  $\tilde{J}_a(z) = J_a(z)$  on any  $U_w \cap U_{w+1}$  for all  $a$ , where  $\tilde{J}_a(z)$  and  $J_a(z)$  are sections of the sheaf of CDO's defined in  $U_w$  and  $U_{w+1}$  respectively. Moreover, from our  $\overline{Q}_+$ -Cech cohomology dictionary, they will be  $\overline{Q}_+$ -closed chiral vertex operators that are holomorphic in  $z$ , which means that one can expand them in a Laurent series that allows an affinisation of the  $G$  Lie-algebra generated by their resulting zero modes. The space of these operators obeys all the physical axioms of a chiral algebra except for reparameterisation invariance on the  $z$ -plane or worldsheet  $\Sigma$ . We will substantiate this last statement next by showing that the holomorphic stress tensor fails to exist in the  $\overline{Q}_+$ -cohomology at the quantum level. Again, this observation will be important in our discussion of a geometric Langlands correspondence for  $G$ .

## 2.3 The Segal-Sugawara tensor and the classical holomorphic chiral algebra

Recall that for any affine algebra  $\widehat{\mathfrak{g}}$  at level  $k \neq -h^\vee$ , one can construct the corresponding stress tensor out of the currents of  $\widehat{\mathfrak{g}}$  via a Segal-Sugawara construction [14]:

$$T(z) = \frac{:d^{ab}J_aJ_b(z):}{k+h^\vee}. \tag{2.8}$$

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<sup>1</sup>Note that one can consistently introduce appropriate fluxes to deform the level away from  $-h^\vee$ —recall from our discussion in section A.7 of [1] that the  $E_{ij} = \partial_i B_j$  term in (2.6) is related to the fluxes that correspond to the moduli of the chiral algebra, and since this term will determine the level  $k$  of the affine  $G$ -algebra via the identification of the global sections  $\tilde{\beta}_i$  with the affine currents valued in the subalgebra of  $\mathfrak{g}$  associated to its positive roots, turning on the relevant fluxes will deform  $k$  away from  $-h^\vee$ . Henceforth, whenever we consider  $k \neq -h^\vee$ , we really mean turning on fluxes in this manner.

As required, for every  $k \neq -h^\vee$ , the modes of the Laurent expansion of  $T(z)$  will span a Virasoro algebra. In particular,  $T(z)$  will generate holomorphic reparametrisations of the coordinates on the worldsheet  $\Sigma$ . Notice that this definition of  $T(z)$  in (2.8) is ill-defined when  $k = -h^\vee$ . Nevertheless, one can always associate  $T(z)$  with the Segal-Sugawara operator  $S(z)$  that is well-defined at any finite level, whereby

$$S(z) = (k + h^\vee)T(z), \tag{2.9}$$

and

$$S(z) = : d^{ab} J_a J_b(z) :. \tag{2.10}$$

From (2.9), we see that  $S(z)$  generates, in its OPE's with other field operators,  $(k + h^\vee)$  times the transformations usually generated by the stress tensor  $T(z)$ . Therefore, at the level  $k = -h^\vee$ ,  $S(z)$  generates no transformations at all — its OPE's with all other field operators are trivial. This is equivalent to saying that the holomorphic stress tensor does not exist at the quantum level, since  $S(z)$ , which is the only well-defined operator at this level that could possibly generate the transformation of fields under an arbitrary holomorphic reparametrisation of the worldsheet coordinates on  $\Sigma$ , acts by zero in the OPE's.

Despite the fact that  $S(z)$  will cease to exist in the spectrum of physical operators associated to the twisted sigma-model on  $X = G/B$  at the quantum level, it will nevertheless exist as a field in its classical  $\overline{Q}_+$ -cohomology or holomorphic chiral algebra. One can convince oneself that this is true as follows. Firstly, from our  $\overline{Q}_+$ -Cech cohomology dictionary, since the  $J_a(z)$ 's are in  $H^0(X, \widehat{\mathcal{O}}_X^{\text{ch}})$ , it will mean that they are in the  $\overline{Q}_+$ -cohomology of the sigma-model at the quantum level. Secondly, since quantum corrections can only annihilate cohomology classes and not create them, it will mean that the  $J_a(z)$ 's will be in the classical  $\overline{Q}_+$ -cohomology of the sigma-model, i.e., the currents are  $\overline{Q}_+$ -closed and are therefore invariant under the transformations generated by  $\overline{Q}_+$  in the absence of quantum corrections. Hence, one can readily see that  $S(z)$  in (2.10) will also be  $\overline{Q}_+$ -closed at the classical level. Lastly, recall from appendix A of [1] that  $[\overline{Q}_+, T(z)] = 0$  such that  $T(z) \neq \{\overline{Q}_+, \dots\}$  in the absence of quantum corrections to the action of  $\overline{Q}_+$  in the classical theory. Note also that the integer  $h^\vee$  in the factor  $(k + h^\vee)$  of the expression  $S(z)$  in (2.9), is due to a shift by  $h^\vee$  in the level  $k$  because of quantum renormalisation effects [15], i.e., the classical expression of  $S(z)$  for a general level  $k$  can actually be written as  $S(z) = kT(z)$ , and therefore, one will have  $[\overline{Q}_+, -h^\vee T(z)] = [\overline{Q}_+, S(z)] = 0$ , where  $S(z) \neq \{\overline{Q}_+, \dots\}$  in the classical theory. Therefore,  $S(z)$  will be a spin-two field in the classical holomorphic chiral algebra of the purely bosonic sector of the twisted sigma-model on  $X = G/B$ . This observation is also consistent with the fact that  $S(z)$  fails to correspond to a global section of the sheaf  $\mathcal{O}_X^{\text{ch}}$  of CDO's — note that in our case, we actually have  $S(z) = -h^\vee T(z)$  in the classical theory, and this will mean that under quantum corrections to the action of  $\overline{Q}_+$ , we will have (see appendix A of [1])  $[\overline{Q}_+, S(z)] = -h^\vee \partial_z (R_{i\bar{j}} \partial_z \phi^i \psi^{\bar{j}}) \neq 0$ , since  $R_{i\bar{j}} \neq 0$  for any flag manifold  $G/B$ . This corresponds in the Cech-cohomology picture to the expression  $\widetilde{S}(z) - \widehat{S}(z) \neq 0$  over an arbitrary intersection  $U_w \cap U_{w+1}$  of open sets, where  $\widetilde{S}(z)$  and  $\widehat{S}(z)$  are sections of the sheaf of CDO's defined in  $U_w$  and  $U_{w+1}$  respectively. This



means that  $\widehat{S}(z)$ , the Čech-cohomology counterpart to the  $S(z)$  operator, will fail to be in  $H^0(X, \widehat{\mathcal{O}}_X^{\text{ch}})$ . Consequently, one can always represent  $S(z)$  by a classical  $c$ -number. This point will be important when we discuss how one can define Hecke eigensheaves that will correspond to flat  ${}^L G$ -bundles on a Riemann surface  $\Sigma$  in our physical interpretation of the geometric Langlands correspondence for  $G$ .

The fact that  $S(z)$  acts trivially in any OPE with other field operators implies that its Laurent modes will commute with the Laurent modes of any of these other field operators; in particular, they will commute with the Laurent modes of the  $J_a(z)$  currents — in other words, the Laurent modes of  $S(z)$  will span the centre  $\mathfrak{z}(\widehat{\mathfrak{g}})$  of the completed universal enveloping algebra of the affine  $G$ -algebra  $\widehat{\mathfrak{g}}$  at the critical level  $k = -h^\vee$  (generated by the Laurent modes of the  $J_a(z)$  currents in the quantum chiral algebra of the twisted sigma-model on  $G/B$  themselves).<sup>2</sup> Notice also that  $S(z)$  is  $\psi^{\bar{j}}$ -independent and is therefore purely bosonic in nature. In other words, the local field  $S(z)$  exists only in the *classical* holomorphic chiral algebra of the  $\psi^{\bar{j}}$ -independent, *purely bosonic* sector of the twisted sigma-model on  $X = G/B$ .

## 2.4 Higher-spin Casimir operators and the classical holomorphic chiral algebra

For an affine  $G$ -algebra, one can generalise the Sugawara formalism to construct higher-spin analogs of the holomorphic stress tensor with the currents. These higher-spin analogs are called Casimir operators, and were first constructed in [16].

In the context of an affine  $G$ -algebra with a module that is furnished by the global sections of the sheaf of CDO's on  $X = G/B$ , a spin- $s_i$  analog of the holomorphic stress tensor will be given by the  $s_i^{\text{th}}$ -order Casimir operator [17]

$$T^{(s_i)}(z) = \frac{: \tilde{d}^{a_1 a_2 a_3 \dots a_{s_i}}(\mathfrak{g}, k)(J_{a_1} J_{a_2} \dots J_{a_{s_i}})(z) :}{k + h^\vee}, \quad (2.11)$$

where  $\tilde{d}^{a_1 a_2 a_3 \dots a_{s_i}}(\mathfrak{g}, k)$  is a completely symmetric traceless  $\mathfrak{g}$ -invariant tensor of rank  $s_i$  that depends on the level  $k$  of the affine  $G$ -algebra. It is also well-defined and finite at  $k = -h^\vee$ . The superscript on  $T^{(s_i)}(z)$  just denotes that it is a spin- $s_i$  analog of  $T(z)$ . Note that  $i = 1, 2, \dots, \text{rank}(\mathfrak{g})$ , and the spins  $s_i$  can take the values  $1 + e_i$ , where  $e_i$ 's are the exponents of  $\mathfrak{g}$ . Thus, one can have  $\text{rank}(\mathfrak{g})$  of these Casimir operators, and the spin-2 Casimir operator is just the holomorphic stress tensor  $T(z)$  from the usual Sugawara construction.

As with  $T(z)$  in (2.8),  $T^{(s_i)}(z)$  is ill-defined when  $k = -h^\vee$ . Nevertheless, one can always make reference to a spin- $s_i$  analog of the Segal-Sugawara tensor  $S^{(s_i)}(z)$  that is well-defined for any finite value of  $k$ , where its relation to  $T^{(s_i)}(z)$  is given by

$$S^{(s_i)}(z) = (k + h^\vee)T^{(s_i)}(z), \quad (2.12)$$

and

$$S^{(s_i)}(z) =: \tilde{d}^{a_1 a_2 a_3 \dots a_{s_i}}(\mathfrak{g}, k)(J_{a_1} J_{a_2} \dots J_{a_{s_i}})(z) :. \quad (2.13)$$

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<sup>2</sup>Notice that  $S(z)$  is constructed out of the currents of the affine  $G$ -algebra by using the invariant tensor  $d^{ab}$  of the corresponding Lie algebra. Consequently, its Laurent modes will span not the centre of the affine algebra, but rather the centre of the completed universal enveloping algebra of the affine algebra.

That is, the operator  $S^{(s_i)}(z)$  generates in its OPE's with all other operators of the quantum theory,  $(k + h^\vee)$  times the field transformations generated by  $T^{(s_i)}(z)$ .

Notice however, that at  $k = -h^\vee$ ,  $S^{(s_i)}(z)$  acts by zero in its OPE with any other operator. This is equivalent to saying that  $T^{(s_i)}(z)$  does not exist as a quantum operator, since the only well-defined operator  $S^{(s_i)}(z)$  which is supposed to generate the field transformations associated to  $T^{(s_i)}(z)$ , act by zero and thus generate no field transformations at all. From our  $\overline{Q}_+$ -Cech cohomology dictionary, this means that the  $\psi^{\bar{j}}$ -independent operator  $T^{(s_i)}(z)$  will fail to correspond to a dimension  $s_i$  global section of  $\widehat{\mathcal{O}}_X^{\text{ch}}$ . Since we have, at the classical level, the relation  $S^{(s_i)}(z) = -h^\vee T^{(s_i)}(z)$ , it will mean that  $S^{(s_i)}(z)$  will also fail to correspond to a dimension  $s_i$  global section of  $\widehat{\mathcal{O}}_X^{\text{ch}}$ . Thus,  $S^{(s_i)}(z)$  will fail to be an operator at the quantum level. Is it even a spin- $s_i$  field in the classical holomorphic chiral algebra of the twisted sigma-model on  $G/B$ , one might ask. The answer is “yes”. To see this, recall that each of the  $J_{a_k}(z)$ 's are separately  $\overline{Q}_+$ -invariant and not  $\overline{Q}_+$ -exact at the classical level. Therefore, the classical counterpart of  $S^{(s_i)}(z)$  in (2.13) must also be such, which in turn means that it will be in the classical  $\overline{Q}_+$ -cohomology and hence classical holomorphic chiral algebra of the twisted sigma-model on  $G/B$ .

The fact that the  $S^{(s_i)}(z)$ 's act trivially in any OPE with other field operators implies that their Laurent modes will commute with the Laurent modes of any other operator; in particular, they will commute with the Laurent modes of the currents  $J_a(z)$  for  $a = 1, 2, \dots, \dim(\mathfrak{g})$ —in other words, the Laurent modes of all rank( $\mathfrak{g}$ ) of the  $S^{(s_i)}(z)$  fields will span fully the centre  $\mathfrak{z}(\widehat{\mathfrak{g}})$  of the completed universal enveloping algebra of  $\widehat{\mathfrak{g}}$  at the critical level  $k = -h^\vee$  (generated by the Laurent modes of the  $J_a(z)$  currents of the quantum chiral algebra of the twisted sigma-model on  $G/B$  themselves). Last but not least, notice that the  $S^{(s_i)}(z)$  fields are also  $\psi^{\bar{j}}$ -independent and are therefore purely bosonic in nature. In other words, the local fields  $S^{(s_i)}(z)$ , for  $i = 1, 2, \dots, \text{rank}(\mathfrak{g})$ —whose Laurent modes will together generate  $\mathfrak{z}(\widehat{\mathfrak{g}})$ —exist only in the *classical* holomorphic chiral algebra of the  $\psi^{\bar{j}}$ -independent, *purely bosonic* sector of the twisted sigma-model on  $X = G/B$ .

## 2.5 The centre $\mathfrak{z}(\widehat{\mathfrak{g}})$ as a Poisson algebra $\mathcal{W}_{-h^\vee}(\widehat{\mathfrak{g}})$

For an affine  $G$ -algebra at an arbitrary level  $k \neq -h^\vee$ , the  $S^{(s_i)}(z)$ 's will exist as  $\psi^{\bar{j}}$ -independent quantum operators in the  $\overline{Q}_+$ -cohomology of the sigma-model. According to our  $\overline{Q}_+$ -Cech cohomology dictionary, the  $S^{(s_i)}(z)$ 's then correspond to classes in  $H^0(X, \widehat{\mathcal{O}}_X^{\text{ch}})$ . Since the cup product of sheaf cohomologies map products of global sections to global sections, it will mean that the OPE of any two  $S^{(s_i)}(z)$  operators must contain another  $S^{(s_i)}(z)$  operator. Moreover, since all the  $S^{(s_i)}(z)$  operators are  $\overline{Q}_+$ -closed, they must form a closed OPE-algebra.<sup>3</sup> What then is this closed OPE-algebra?

To answer this, first recall that for some  $k \neq -h^\vee$ , the  $S^{(s_i)}(z)$ 's have a quantum definition whereby  $S^{(s_i)}(z) = (k + h^\vee)T^{(s_i)}(z)$ . The Casimir operators  $T^{(s_i)}(z)$  are known to span (up to null or  $\overline{Q}_+$ -exact operators in our interpretation) a closed  $\mathcal{W}$  OPE-algebra

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<sup>3</sup>Note that if  $\mathcal{O}$  and  $\mathcal{O}'$  are non-exact  $\overline{Q}_+$ -closed observables in the  $\overline{Q}_+$ -cohomology, i.e.,  $\{\overline{Q}_+, \mathcal{O}\} = \{\overline{Q}_+, \mathcal{O}'\} = 0$ , then  $\{\overline{Q}_+, \mathcal{O}\mathcal{O}'\} = 0$ . Moreover, if  $\{\overline{Q}_+, \mathcal{O}\} = 0$ , then  $\mathcal{O}\{\overline{Q}_+, W\} = \{\overline{Q}_+, \mathcal{O}W\}$  for any observable  $W$ . These two statements mean that the cohomology classes of observables that commute with  $\overline{Q}_+$  form a closed and well-defined algebra.

associated to  $\widehat{\mathfrak{g}}$  [17]. Since the spin-2 Casimir operator  $T^{(2)}(z)$  generates a Virasoro OPE-algebra of central charge  $c = k \dim(\mathfrak{g})/(k + h^\vee)$ , the  $S^{(s_i)}(z)$ 's will then span a rescaled (by a factor of  $(k + h^\vee)$ ) version of the closed  $\mathcal{W}$  OPE-algebra associated to  $\widehat{\mathfrak{g}}$  of central charge  $c = k \dim(\mathfrak{g})/(k + h^\vee)$  for  $k \neq -h^\vee$ .

Since each  $S^{(s_i)}(z)$  is holomorphic in  $z$ , we can Laurent expand it as

$$S^{(s_i)}(z) = \sum_{n \in \mathbb{Z}} \hat{S}_n^{(s_i)} z^{-n-s_i}. \tag{2.14}$$

Let us henceforth denote  $\mathcal{W}_k(\widehat{\mathfrak{g}})$  to be the closed algebra generated by the Laurent modes  $\hat{S}_n^{(s_i)}$  where  $k \neq -h^\vee$ . At  $k \neq -h^\vee$ , since  $S^{(2)}(z) = (k + h^\vee)T(z)$ , the Laurent modes  $\hat{S}_n^{(2)}$  must then generate the Virasoro algebra with the following quantum commutator relations:

$$[\hat{S}_n^{(2)}, \hat{S}_m^{(2)}] = (k + h^\vee) \left( (n - m) \hat{S}_{n+m}^{(2)} + \frac{k \dim(\mathfrak{g})}{12} (n^3 - n) \delta_{n,-m} \right). \tag{2.15}$$

Likewise, the other quantum commutator relations spanned by the Laurent modes of the other spin- $s_i$  operators, will take the same form as (2.15) and have a factor of  $(k + h^\vee)$  in front. Since we will have no need to refer to these explicit relations in our discussions, we shall omit them for brevity, as they can get rather complicated very quickly.

Now, let us consider the case when  $k = -h^\vee$ . From our earlier explanations about the nature of the  $S^{(s_i)}(z)$  operators, we find that they will cease to exist as quantum operators at this critical level. Since we understand that the  $S^{(s_i)}(z)$ 's must be holomorphic classical fields at  $k = -h^\vee$ , we shall rewrite the Laurent expansion of  $S^{(s_i)}(z)$  as

$$S^{(s_i)}(z) = \sum_{n \in \mathbb{Z}} S_n^{(s_i)} z^{-n-s_i}, \tag{2.16}$$

so as to differentiate the classical modes of expansion  $S_n^{(s_i)}$  from their quantum counterparts  $\hat{S}_n^{(s_i)}$  in (2.14). Unlike the  $\hat{S}_n^{(s_i)}$ 's which obey the quantum commutator relations of a  $\mathcal{W}_k(\widehat{\mathfrak{g}})$ -algebra for an arbitrary level  $k \neq -h^\vee$ , the  $S_n^{(s_i)}$ 's, being the modes of a Laurent expansion of a classical field, will instead obey Poisson bracket relations that define a certain classical algebra at  $k = -h^\vee$ . Since every  $\hat{S}_n^{(s_i)}$  must reduce to its classical counterpart  $S_n^{(s_i)}$  at  $k = -h^\vee$ , one can see that by taking  $(k + h^\vee) \rightarrow 0$ , we are actually going to the classical limit. This is analogous to taking the  $\hbar \rightarrow 0$  limit in any quantum mechanical theory whenever one wants to ascertain its classical counterpart. In fact, by identifying  $(k + h^\vee)$  with  $i\hbar$ , and by noting that one must make the replacement from Poisson brackets to commutators — that is,  $\{E_n^{(s_i)}, E_m^{(s_j)}\}_{P.B.} \rightarrow \frac{1}{i\hbar} [\hat{E}_n^{(s_i)}, \hat{E}_m^{(s_j)}]$  — in quantising any classical mode  $E_n^{(s_i)}$  into an operator  $\hat{E}_n^{(s_i)}$ , we can ascertain the classical algebra generated by the  $S_n^{(s_i)}$ 's from the  $\mathcal{W}_k(\widehat{\mathfrak{g}})$ -algebra commutator relations that their quantum counterparts — the  $\hat{S}_n^{(s_i)}$ 's — satisfy. Since all the  $S^{(s_i)}(z)$  fields must now lie in the classical  $\overline{\mathcal{Q}}_+$ -cohomology of the twisted sigma-model on  $G/B$ , it will mean that their Laurent modes  $S_n^{(s_i)}$  must also generate a closed, classical algebra associated to  $\widehat{\mathfrak{g}}$ , which, we shall henceforth denote as  $\mathcal{W}_{-h^\vee}(\widehat{\mathfrak{g}})$ . In order to ascertain the central charge of this *classical*  $\mathcal{W}_{-h^\vee}(\widehat{\mathfrak{g}})$ -algebra, it suffices to determine the central charge of its classical Virasoro subalgebra generated by

the  $S_m^{(2)}$ 's. From (2.15), we find that as  $k \rightarrow -h^\vee$ , the  $S_m^{(2)}$ 's satisfy

$$\{S_n^{(2)}, S_m^{(2)}\}_{P.B.} = (n - m)S_{n+m}^{(2)} - \frac{h^\vee \dim(\mathfrak{g})}{12} (n^3 - n) \delta_{n,-m}, \quad (2.17)$$

the classical Virasoro algebra with central charge  $c = -h^\vee \dim(\mathfrak{g})$ . Hence, the  $S_n^{(s_i)}$ 's will generate a classical  $\mathcal{W}_{-h^\vee}(\widehat{\mathfrak{g}})$ -algebra of central charge  $c = -h^\vee \dim(\mathfrak{g})$ . For example, the specific case of  $\mathfrak{g} = \mathfrak{sl}_2$  was considered in section 2.1 of [1]—the modes  $S_m^{(2)}$  were found to generate a classical  $\mathcal{W}_{-h^\vee}(\widehat{\mathfrak{sl}}_2)$ -algebra with central charge  $c = -h^\vee \dim(\mathfrak{sl}_2) = -6$ , where  $h^\vee = 2$  and  $\dim(\mathfrak{sl}_2) = 3$ . The specific case of  $\mathfrak{g} = \mathfrak{sl}_3$  was also considered in section 2.3 of [1]—the modes  $S_m^{(2)}$  and  $S_m^{(3)}$  were found to generate a classical  $\mathcal{W}_{-h^\vee}(\widehat{\mathfrak{sl}}_3)$ -algebra with central charge  $c = -h^\vee \dim(\mathfrak{sl}_3) = -24$ , where  $h^\vee = 3$  and  $\dim(\mathfrak{sl}_3) = 8$ .

Recall at this point that the Laurent modes of the  $S^{(s_i)}(z)$  fields for  $i = 1, 2, \dots, \text{rank}(\mathfrak{g})$ , will together generate  $\mathfrak{z}(\widehat{\mathfrak{g}})$ , the centre of the completed universal enveloping algebra of the affine  $G$ -algebra  $\widehat{\mathfrak{g}}$  at the critical level  $k = -h^\vee$ . Hence, we have an identification of Poisson algebras  $\mathfrak{z}(\widehat{\mathfrak{g}}) \simeq \mathcal{W}_{-h^\vee}(\widehat{\mathfrak{g}})$ .

Last but not least, another way to understand why  $\mathfrak{z}(\widehat{\mathfrak{g}})$  must be a classical (or Poisson) algebra is as follows. Firstly, let us consider the general case of  $k \neq -h^\vee$ , whereby the  $\hat{S}_n^{(2)}$  modes can be related to the  $J_n^a$  modes of  $\widehat{\mathfrak{g}}$  via the quantum commutator relations

$$[\hat{S}_n^{(2)}, J_m^a] = -(k + h^\vee)mJ_{n+m}^a, \quad (2.18)$$

$$[\hat{S}_n^{(2)}, \hat{S}_m^{(2)}] = (k + h^\vee) \left( (n - m)\hat{S}_{n+m}^{(2)} + \frac{k}{12} \dim(\mathfrak{g}) (n^3 - n) \delta_{n,-m} \right), \quad (2.19)$$

where  $a = 1, 2, \dots, \dim(\mathfrak{g})$ . If we now let  $k = -h^\vee$ , we will have  $[\hat{S}_n^{(2)}, J_m^a] = [\hat{S}_n^{(2)}, \hat{S}_m^{(2)}] = 0$ . Hence, one can define simultaneous eigenstates of the  $\hat{S}_n^{(2)}$  and  $J_n^a$  mode operators. In particular, one would be able to properly define a general state  $\Psi = \hat{S}_{-l}^{(2)} \hat{S}_{-q}^{(2)} \dots \hat{S}_{-p}^{(2)} |0, \alpha\rangle$ , where  $|0, \alpha\rangle$  is a vacuum state associated to a representation of  $\mathfrak{g}$  labelled by  $\alpha$ , such that  $J_0^a |0, \alpha\rangle = \alpha^a |0, \alpha\rangle$ . However, note that any such  $\Psi$  will correspond to a null-state, i.e.,  $\Psi$  decouples from the real, physical Hilbert space of quantum states spanned by the representations of  $\mathfrak{g}$  [18]. This means that the  $\hat{S}_m^{(2)}$ 's which span  $\mathfrak{z}(\widehat{\mathfrak{g}})$  cannot exist as quantum mode operators. Hence, since  $\mathfrak{z}(\widehat{\mathfrak{g}})$  must be closed in all the  $\hat{S}_m^{(s_i)}$  modes, it must therefore be a classical algebra at  $k = -h^\vee$ .

### 3. The $B$ -gauged WZW model on $G$ and the $\mathcal{W}$ -algebra $\mathcal{W}_{k'}(\widehat{\mathfrak{g}})$

In this section, we shall explain how a *dual* description of the holomorphic chiral algebra of the purely bosonic sector of the sigma-model on  $G/B$ , can be given by the *holomorphic* BRST-cohomology (or chiral algebra) of a  $B$ -gauged WZW model on  $G$ . We then proceed to outline the mathematical Drinfeld-Sokolov reduction procedure [9] of generating the Hecke algebra of local operators which defines  $\mathcal{W}_{k'}(\widehat{\mathfrak{g}})$ —the  $\mathcal{W}$ -algebra associated to  $\widehat{\mathfrak{g}}$  at level  $k'$ . Lastly, we will show that the holomorphic BRST-cohomology of the  $B$ -gauged WZW model on  $G$  at level  $k'$  physically realises, in all generality, this Hecke algebra of local operators.

### 3.1 A dual description of the purely bosonic holomorphic chiral algebra

Let us now seek a dual description of the above classical, holomorphic chiral algebra of the twisted sigma-model on  $G/B$  spanned by the  $S^{(s_i)}(z)$ 's. To this end, let us first generalise the action of the twisted sigma-model by making the replacement  $g_{i\bar{j}} \rightarrow g_{i\bar{j}} + b_{i\bar{j}}$  in  $V$  of  $S_{twist}$  of (A.9) in [1], where  $b_{i\bar{j}}$  is a  $(1,1)$ -form on the target space  $X$  associated to a B-field. This just adds to  $S_{twist}$  a *cohomologically-trivial*  $\bar{Q}_+$ -exact term  $\{\bar{Q}_+, -b_{i\bar{j}}\psi_{\bar{z}}^i \partial_z \phi^{\bar{j}}\}$ , and does nothing to change our above discussions about the chiral algebra of the sigma-model. This generalised action can be explicitly written as

$$S_{gen} = \int_{\Sigma} |d^2z| (g_{i\bar{j}} + b_{i\bar{j}})(\partial_z \phi^{\bar{j}} \partial_{\bar{z}} \phi^i) + g_{i\bar{j}} \psi_{\bar{z}}^i D_z \psi^{\bar{j}} + b_{i\bar{j}} \psi_{\bar{z}}^i \partial_z \psi^{\bar{j}} + b_{i\bar{j}} \psi_{\bar{z}}^i \partial_z \phi^{\bar{j}} \psi^{\bar{j}}. \quad (3.1)$$

Now recall that the  $S^{(s_i)}(z)$ 's exist in the classical holomorphic chiral algebra of the  $\psi^{\bar{j}}$ -independent sector of the twisted sigma-model on  $G/B$ . This means that in order for one to ascertain the dual description of the  $S^{(s_i)}(z)$ 's, it suffices to confine oneself to the study of the holomorphic chiral algebra of the  $\psi^{\bar{j}}$ -independent, purely bosonic sector of the twisted sigma-model on  $G/B$ . The  $\psi^{\bar{j}}$ -independent specialisation of  $S_{gen}$ , which describes this particular sector of interest, can be written as

$$S_{bosonic} = \int_{\Sigma} |d^2z| (g_{i\bar{j}} + b_{i\bar{j}}) \partial_z \phi^i \partial_{\bar{z}} \phi^{\bar{j}}. \quad (3.2)$$

Notice that  $S_{bosonic}$  just describes a non-linear sigma-model of a free bosonic string which propagates in a  $G/B$  target-space. Note that a non-linear sigma-model on any homogenous coset space such as  $G/B$ , can be described by an *asymmetrically*  $B$ -gauged WZW model on  $G$  that is associated with the action  $g \rightarrow gb^{-1}$ , where  $g \in G$  and  $b \in B$ . However, upon a BRST-quantisation, one can easily see that the BRST-cohomology of the asymmetrically  $B$ -gauged WZW model on  $G$ , coincides exactly with the *holomorphic* (or left-moving) sector of the total BRST-cohomology of a *symmetrically*  $B$ -gauged WZW model on  $G$  that is associated with the action  $g \rightarrow bgb^{-1}$ . In other words, at the level of the holomorphic chiral algebra, a physically equivalent description of the  $\psi^{\bar{j}}$ -independent, non-supersymmetric sector of the twisted sigma-model on  $G/B$ , will be given by a symmetrically  $B$ -gauged WZW model on  $G$  that is genuinely gauge-invariant on the worldsheet  $\Sigma$ .<sup>4</sup> In other words, the  $S^{(s_i)}(z)$ 's should correspond to observables in the classical holomorphic BRST-cohomology of the  $B$ -gauged WZW model on  $G$ .

### 3.2 The $B$ -gauged WZW model on $G$

let us now proceed to describe the relevant  $B$ -gauged WZW model on  $G$  in detail.<sup>5</sup> First,

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<sup>4</sup>Henceforth, whenever we refer to the  $B$ -gauged WZW model on  $G$ , we really mean the symmetrically gauged WZW model on  $G$  that is genuinely gauge-invariant on the worldsheet  $\Sigma$ .

<sup>5</sup>It may be disconcerting to some readers at this point that the Borel subgroup  $B$  which we are gauging the  $G$  WZW model by, is non-compact in general. Apart from citing several well-known examples in the physics literature [19–23] that have done likewise to consider non-compact WZW models gauged to non-compact (sometimes Borel) subgroups, one can also argue that our model is actually equivalent — within our context — to a physically consistent model which gauges a *compact* subgroup instead. Firstly, note

note that the action of the most general WZW model can be written as

$$S_{\text{WZ}}(g) = \frac{k'}{4\pi} \int_{\Sigma} d^2z \operatorname{Tr}(\partial_z g^{-1} \partial_{\bar{z}} g) + \frac{ik'}{24\pi} \int_{B; \partial B = \Sigma} d^3x \operatorname{Tr}(g^{-1} dg)^3, \quad (3.3)$$

where  $k'$  is the level, and  $g$  is a worldsheet scalar field valued in any connected Lie group  $G$  that is also periodic along one of the worldsheet directions with period  $2\pi$ .<sup>6</sup> The trace  $\operatorname{Tr}$  is the usual matrix trace in the defining representation of  $G$ .

A gauged version of (3.3) can be written as

$$S_{\text{gauged}}(g, A_z, A_{\bar{z}}) = S_{\text{WZ}}(g) + \frac{k'}{2\pi} \int_{\Sigma} d^2z \operatorname{Tr}[A_z(\partial_z g g^{-1} + \bar{M}) - A_{\bar{z}}(g^{-1} \partial_z g + M) + A_z g A_{\bar{z}} g^{-1} - A_z A_{\bar{z}}], \quad (3.4)$$

where the worldsheet one-form gauge field  $A = A_z dz + A_{\bar{z}} d\bar{z}$  is valued in  $\mathfrak{h}$ , the Lie algebra of a subgroup  $H$  of  $G$ . Notice that  $S_{\text{gauged}}(g, A_z, A_{\bar{z}})$  differs slightly from the standard form of a gauged WZW model commonly found in the physical literature — additional  $\bar{M}$  and  $M$  constant matrices have been incorporated in the  $\partial_z g g^{-1}$  and  $g^{-1} \partial_z g$  terms of the standard action, so that one can later use them to derive the correct form of the holomorphic stress tensor without reference to a coset formalism. Setting  $\bar{M}$  and  $M$  to the zero matrices simply takes us back to the standard action for the gauged WZW model. As required,  $S_{\text{gauged}}(g, A_z, A_{\bar{z}})$  is invariant under the standard (chiral) local gauge transformations

$$g \rightarrow h g h^{-1}; \quad A_z \rightarrow \partial_z h \cdot h^{-1} + h A_z h^{-1}; \quad A_{\bar{z}} \rightarrow \partial_{\bar{z}} h \cdot h^{-1} + h A_{\bar{z}} h^{-1}, \quad (3.5)$$

where  $h = e^{\lambda(z, \bar{z})} \in H$  for any  $\lambda(z, \bar{z}) \in \mathfrak{h}$ .<sup>7</sup> The invariance of (3.4) under the gauge transformations in (3.5) can be verified as follows. Firstly, note that the  $\bar{M}(M)$ -independent terms make up the usual Lagrangian for the standard gauged WZW action, which is certainly

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that for a complex flag manifold  $G/B$ , we have the relation  $G/B = \mathcal{G}/\mathcal{T}$ , where  $\mathcal{G}$  is a compact group whose Lie algebra is the real form of the Lie algebra of  $G$ , and  $\mathcal{T}$  is the maximal torus of purely diagonal matrices in  $\mathcal{G}$  [24]—in other words,  $\mathcal{T}$  is an anomaly-free, *compact* diagonal subgroup in the context of a  $\mathcal{T}$ -gauged WZW model on  $\mathcal{G}$ . Secondly, note that the OPE algebras of the affine  $G$ -algebra and the affine  $\mathcal{G}$ -algebra are the same. These two points imply that at the level of their holomorphic BRST-cohomologies, the  $B$ -gauged WZW model on  $G$  is equivalent to the  $\mathcal{T}$ -gauged WZW model on  $\mathcal{G}$  that can always be physically consistently defined, and whose gauge group is also compact. However, since one of our main aims in this paper is to relate the gauged WZW model to the algebraic DS-reduction scheme, we want to consider the  $B$ -gauged WZW model on  $G$ . Last but not least, note that we will ultimately be interested only in the *classical* spectrum of the gauged WZW model, whereby the compactness or non-compactness of the gauge group will be irrelevant.

<sup>6</sup>Note that in some situations, the target group manifold of the WZW model is not simply-connected; the complex  $D$ -group or  $\text{SO}(N, \mathbf{C})$  manifold is one such example. In this case, the non-simple-connectedness of the group will translate to a restriction in the values that  $k'$  can take [25]. In other situations, one must exclude some representations and include winding sectors in the Hilbert space of states. However, since our results will only depend on the classical spectrum of local fields of the WZW model in the limit  $k' \rightarrow \infty$ , we can, for our purpose, ignore this technical subtlety.

<sup>7</sup>A similar model has been considered in [23]. However, the action in that context is instead invariant under a *non-chiral* local gauge transformation. Moreover, it does not contain the  $A_z A_{\bar{z}}$  term present in a standard gauged WZW model.

invariant under the gauge transformations of (3.5). Next, note that under an infinitesimal gauge transformation  $h \simeq 1 + \lambda$ , the terms  $\text{Tr}(A_z \bar{M})$  and  $\text{Tr}(A_{\bar{z}} M)$  change as

$$\delta \text{Tr}(A_z \bar{M}) = \text{Tr}(\partial_z \lambda \bar{M}) - \text{Tr}(\bar{M} [\lambda, A_z]), \quad (3.6)$$

$$\delta \text{Tr}(A_{\bar{z}} M) = \text{Tr}(\partial_{\bar{z}} \lambda M) - \text{Tr}(M [\lambda, A_{\bar{z}}]). \quad (3.7)$$

Since we will be considering the case where  $H$  is the Borel subgroup of  $G$  and therefore,  $\lambda$  and  $A$  will be valued in the Lie algebra of a maximally *solvable* (Borel) subgroup of  $G$ , the second term on the r.h.s. of (3.6) and (3.7) will be zero [23]. What remains are total divergence terms that will vanish upon integration on  $\Sigma$  because it is a worldsheet with no boundaries. Therefore, unless  $H$  is a Borel subgroup of  $G$  (or any other solvable subgroup of  $G$ ), one cannot incorporate  $\bar{M}$  and  $M$  in the action and still maintain the requisite gauge invariance. This explains why generalisations of gauged WZW models with these constant matrices  $\bar{M}$  and  $M$  have not appeared much in the physical literature. Nevertheless, this generalisation can be considered in our case. As we shall see shortly, this generalisation will allow us to obtain the correct form of the holomorphic stress tensor of the  $B$ -gauged WZW model on  $G$  without any explicit reference to a coset formalism.

The classical equations of motion that follow from the field variations in (3.5) are

$$\delta A_z : \quad D_{\bar{z}} g g^{-1}|_H = -M_+, \quad (3.8)$$

$$\delta A_{\bar{z}} : \quad g^{-1} D_z g|_H = -M_-, \quad (3.9)$$

$$\delta g : \quad D_{\bar{z}}(g^{-1} D_z g) = F_{z\bar{z}}, \quad (3.10)$$

$$\delta g : \quad D_z(D_{\bar{z}} g g^{-1}) = F_{\bar{z}z}, \quad (3.11)$$

where  $F_{z\bar{z}} = \partial_z A_{\bar{z}} - \partial_{\bar{z}} A_z + [A_z, A_{\bar{z}}]$  and  $F_{\bar{z}z} = \partial_{\bar{z}} A_z - \partial_z A_{\bar{z}} + [A_{\bar{z}}, A_z]$  are the non-vanishing components of the field strength, and the covariant derivatives are given by  $D_z = \partial_z + [A_z, \ ]$  and  $D_{\bar{z}} = \partial_{\bar{z}} + [A_{\bar{z}}, \ ]$ . By imposing the condition of (3.9) in (3.10), and by imposing the condition of (3.8) in (3.11), since  $M_{\pm}$  are constant matrices, we find that we have the zero curvature condition  $F_{z\bar{z}} = F_{\bar{z}z} = 0$  as expected of a non-dynamically gauged WZW model. This means that  $A_z$  and  $A_{\bar{z}}$  are trivial on-shell. One is then free to use the gauge invariance to set  $A_z$  and/or  $A_{\bar{z}}$  to a constant such as zero. In setting  $A_z = A_{\bar{z}} = 0$  in (3.10) and (3.11), noting that  $F_{z\bar{z}} = F_{\bar{z}z} = 0$ , we have the relations

$$\partial_{\bar{z}}(g^{-1} \partial_z g) = 0 \quad \text{and} \quad \partial_z(\partial_{\bar{z}} g g^{-1}) = 0. \quad (3.12)$$

In other words, we have a  $\mathfrak{g}$ -valued, holomorphic conserved current  $J(z) = g^{-1} \partial_z g$ , and a  $\mathfrak{g}$ -valued antiholomorphic conserved current  $\bar{J}(\bar{z}) = \partial_{\bar{z}} g g^{-1}$ , both of which are dimension one and generate affine symmetries on  $\Sigma$ . The action in (3.4) can thus be written as

$$S_{\text{gauged}}(g, A_z, A_{\bar{z}}) = S_{\text{WZ}}(g) + \frac{k'}{2\pi} \int_{\Sigma} d^2z \text{Tr}[A_z(\bar{J}(\bar{z}) + \bar{M}) - A_{\bar{z}}(J(z) + M) + A_z g A_{\bar{z}} g^{-1} - A_z A_{\bar{z}}]. \quad (3.13)$$

For our case where  $H$  is a Borel subgroup  $B$  of  $G$ , one can further simplify (3.13) as follows. Firstly, since  $G$  is a connected group, its Lie algebra  $\mathfrak{g}$  will have a Cartan decomposition  $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{c} \oplus \mathfrak{n}_+$ , where  $\mathfrak{c}$  is the Cartan subalgebra, and  $\mathfrak{n}_{\pm}$  are

the nilpotent subalgebras of the the upper and lower triangular matrices of  $G$ . The Borel subalgebras will then be given by  $\mathfrak{b}_\pm = \mathfrak{c} \oplus \mathfrak{n}_\pm$ , and they correspond to the Borel subgroups  $B_\pm$ . For the complex flag manifolds that we will be considering in this paper,  $B_+$  will be the Borel subgroup of interest.  $B$  will henceforth mean  $B_+$  in all of our proceeding discussions. With respect to this decomposition of the Lie algebra  $\mathfrak{g}$ , we can write  $J(z) = \sum_{a=1}^{\dim n^-} J_-^a(z)t_a^- + \sum_{a=1}^{\dim \mathfrak{c}} J_c^a(z)t_a^c + \sum_{a=1}^{\dim n^+} J_+^a(z)t_a^+$ , and  $\bar{J}(\bar{z}) = \sum_{a=1}^{\dim n^-} \bar{J}_-^a(\bar{z})t_a^- + \sum_{a=1}^{\dim \mathfrak{c}} \bar{J}_c^a(\bar{z})t_a^c + \sum_{a=1}^{\dim n^+} \bar{J}_+^a(\bar{z})t_a^+$ , where  $t_a^- \in \mathfrak{n}_-$ ,  $t_a^c \in \mathfrak{c}$ , and  $t_a^+ \in \mathfrak{n}_+$ . One can also write  $M = \sum_{a=1}^{\dim n^-} M_-^a t_a^- + \sum_{a=1}^{\dim \mathfrak{c}} M_c^a t_a^c + \sum_{a=1}^{\dim n^+} M_+^a t_a^+$ , and  $\bar{M} = \sum_{a=1}^{\dim n^-} \bar{M}_-^a t_a^- + \sum_{a=1}^{\dim \mathfrak{c}} \bar{M}_c^a t_a^c + \sum_{a=1}^{\dim n^+} \bar{M}_+^a t_a^+$ , where  $M_{\pm;c}^a (\bar{M}_{\pm;c}^a)$  are arbitrary number constants. Next, note that  $H = B$ , and  $B \simeq N_+$ , where  $N_+ = [B, B]$  is the subgroup of  $G$  generated by its Lie algebra  $\mathfrak{n}_+$  of strictly upper triangular matrices which are traceless, i.e., for  $t, t' \in \mathfrak{n}_+$ , we have  $\text{Tr}_L(tt') - \text{Tr}_R(tt') = 0$ , where the trace  $\text{Tr}_L$  and  $\text{Tr}_R$  are taken over some  $L$  and  $R$  representation of  $G$  respectively. In other words,  $N_+$  is the non-anomalous subgroup to be gauged, and we can write  $A_z = \sum_{a=1}^{\dim n^+} \tilde{A}_z^a t_a^+$ , and  $A_{\bar{z}} = \sum_{a=1}^{\dim n^+} \tilde{A}_{\bar{z}}^a t_a^+$ . Next, note that since  $\text{Tr}(t_a^\alpha t_b^\beta) = \delta_{a,b} \delta^{\alpha,\beta}$ , the trace of the second term on the r.h.s. of (3.13) will be non-vanishing only for components of  $J(z)(\bar{J}(\bar{z}))$  and  $M(\bar{M})$  that are associated to their expansion in  $\mathfrak{n}_+$ . Let us denote  $J^+(z) = \sum_{a=1}^{\dim n^+} J_+^a(z)t_a^+$  and  $M^+ = \sum_{a=1}^{\dim n^+} M_+^a t_a^+$ . Let us also denote  $\bar{J}^+(\bar{z}) = \sum_{a=1}^{\dim n^+} \bar{J}_+^a(\bar{z})t_a^+$  and  $\bar{M}^+ = \sum_{a=1}^{\dim n^+} \bar{M}_+^a t_a^+$ . Then, one can write the action of a  $B$ -gauged WZW model on  $G$  as

$$\begin{aligned}
 S_{B\text{-gauged}}(g, A_z, A_{\bar{z}}, J^+, \bar{J}^+) \\
 = S_{\text{WZ}}(g) - \frac{k'}{2\pi} \int_{\Sigma} d^2 z \sum_{l=1}^{\dim n^+} \left[ \tilde{A}_{\bar{z}}^l (J_+^l(z) + M_+^l) - \tilde{A}_z^l (\bar{J}_+^l(\bar{z}) + \bar{M}_+^l) \right] \\
 - \text{Tr}[A_z g A_{\bar{z}} g^{-1} - A_z A_{\bar{z}}]. \tag{3.14}
 \end{aligned}$$

Due to the  $B$ -gauge invariance of the theory, we must divide the measure in any path integral computation by the volume of the  $B$ -gauge symmetry. That is, the partition function has to take the form

$$Z_G = \int_{\Sigma} \frac{[g^{-1} dg, d\tilde{A}_z^l, d\tilde{A}_{\bar{z}}^l]}{(\text{gauge volume})} \exp(i S_G(g, A_z, A_{\bar{z}}, J^+, \bar{J}^+)). \tag{3.15}$$

One must now fix this gauge invariance to eliminate the non-unique degrees of freedom. One can do this by employing the BRST formalism which requires the introduction of Faddeev-Popov ghost fields. In order to obtain the *holomorphic* BRST transformations of the fields, one simply replaces the infinitesimal position-dependent parameters  $\epsilon^l$  of  $h = \exp(-\sum_{l=1}^{\dim n^+} \epsilon^l t_m^+) \in B$  in the corresponding *left-sector* of the gauge transformations in (3.5) with the ghost fields  $c^l$ , which then gives us

$$\delta_{\text{BRST}}(g) = -c^l t_l^+ g, \quad \delta_{\text{BRST}}(\tilde{A}_{\bar{z}}^l) = -D_{\bar{z}} c^l, \quad \delta_{\text{BRST}}(\text{others}) = 0. \tag{3.16}$$

The components of the ghost field  $c(z) = \sum_{l=1}^{\dim n^+} c^l(z)t_l^+$  and those of its anti-ghost partner  $b(z) = \sum_{l=1}^{\dim n^+} b^l(z)t_l^+$  will transform as

$$\delta_{\text{BRST}}(c^l) = -\frac{1}{2} f_{mk}^l c^m c^k, \quad \delta_{\text{BRST}}(b^l) = \tilde{B}^l, \quad \delta_{\text{BRST}}(\tilde{B}^l) = 0, \tag{3.17}$$



where the  $f_{mk}^l$ 's are the structure constants of the nilpotent subalgebra  $\mathfrak{n}_+$ . Also, the  $\tilde{B}^l$ 's are the Nakanishi-Lautrup auxiliary fields that are the BRST transforms of the  $b^l$ 's. They also serve as Lagrange multipliers to impose the gauge-fixing conditions.

In order to obtain the *antiholomorphic* BRST transformations of the fields, one employs the same recipe with the corresponding *right-sector* of the gauge transformations in (3.5), with the infinitesimal position-dependent gauge parameter now replaced by the ghost field  $\bar{c}^l$ , which then gives us

$$\bar{\delta}_{\text{BRST}}(g) = \bar{c}^l t_l^+ g, \quad \bar{\delta}_{\text{BRST}}(\tilde{A}_z^l) = -D_z \bar{c}^l, \quad \bar{\delta}_{\text{BRST}}(\text{others}) = 0. \quad (3.18)$$

The components of the ghost field  $\bar{c}(\bar{z}) = \sum_{l=1}^{\dim \mathfrak{n}_+} \bar{c}^l(\bar{z}) t_l^+$  and those of its anti-ghost partner  $\bar{b}(\bar{z}) = \sum_{l=1}^{\dim \mathfrak{n}_+} \bar{b}^l(\bar{z}) t_l^+$  will transform as

$$\bar{\delta}_{\text{BRST}}(\bar{c}^l) = -\frac{1}{2} f_{mk}^l \bar{c}^m \bar{c}^k, \quad \bar{\delta}_{\text{BRST}}(\bar{b}^l) = \tilde{B}^l, \quad \bar{\delta}_{\text{BRST}}(\tilde{B}^l) = 0. \quad (3.19)$$

In the above, the  $\tilde{B}^l$ 's are the Nakanishi-Lautrup auxiliary fields that are the antiholomorphic BRST transforms of the  $\bar{b}^l$  fields. They also serve as Lagrange multipliers to impose the gauge-fixing conditions.

Since the BRST transformations in (3.16) and (3.18) are just infinitesimal versions of the gauge transformations in (3.5),  $S_{B\text{-gauged}}(g, A_z, A_{\bar{z}}, J^+, \bar{J}^+)$  will be invariant under them. An important point to note is that in addition to  $(\delta_{\text{BRST}} + \bar{\delta}_{\text{BRST}}) \cdot (\delta_{\text{BRST}} + \bar{\delta}_{\text{BRST}}) = 0$ , the holomorphic and antiholomorphic BRST-variations are also separately nilpotent, i.e.,  $\delta_{\text{BRST}}^2 = 0$  and  $\bar{\delta}_{\text{BRST}}^2 = 0$ . Moreover,  $\delta_{\text{BRST}} \cdot \bar{\delta}_{\text{BRST}} = -\bar{\delta}_{\text{BRST}} \cdot \delta_{\text{BRST}}$ . This means that the BRST-cohomology of the  $B$ -gauged WZW model on  $G$  can be decomposed into *independent* holomorphic and antiholomorphic sectors that are just complex conjugate of each other, and that it can be computed via a spectral sequence, whereby the first two complexes will be furnished by its holomorphic and antiholomorphic BRST-cohomologies respectively. Since we will only be interested in the holomorphic chiral algebra of the  $B$ -gauged WZW model on  $G$  (which, by the way, is just identical to its antiholomorphic chiral algebra by a complex conjugation as mentioned), we shall henceforth focus on the *holomorphic* BRST-cohomology of the  $B$ -gauged WZW model on  $G$ .

By the usual recipe of the BRST formalism, one can fix the gauge by adding to the BRST-invariant action  $S_{B\text{-gauged}}(g, A_z, A_{\bar{z}}, J^+, \bar{J}^+)$ , a BRST-exact term. Since the BRST transformation by  $(\delta_{\text{BRST}} + \bar{\delta}_{\text{BRST}})$  is nilpotent, the new total action will still be BRST-invariant as required. The choice of the BRST-exact operator will then define the gauge-fixing conditions. A consistent choice of the BRST-exact operator that will give us the requisite action for the ghost and anti-ghost fields is

$$S_{B\text{-gauged}}(g, A_z, A_{\bar{z}}, J^+, \bar{J}^+) + (\delta_{\text{BRST}} + \bar{\delta}_{\text{BRST}}) \left( \frac{k'}{2\pi} \int_{\Sigma} d^2 z \sum_{l=1}^{\dim \mathfrak{n}_+} \tilde{A}_z^l b^l + \tilde{A}_z^l \bar{b}^l \right),$$

where one will indeed have the desired total action, which can be written as

$$\begin{aligned}
 S_{\text{WZW}}(g) &= \frac{k'}{2\pi} \int_{\Sigma} d^2z \left\{ \sum_{l=1}^{\dim n_+} \left[ \tilde{A}_{\bar{z}}^l (J_+^l(z) + M_+^l - \tilde{B}^l) - \tilde{A}_z^l (\bar{J}_+^l(\bar{z}) + \bar{M}_+^l + \tilde{\bar{B}}^l) \right] \right. \\
 &\quad \left. - \text{Tr}[A_z g A_{\bar{z}} g^{-1} - A_z A_{\bar{z}}] \right\} \\
 &+ \frac{k'}{2\pi} \int_{\Sigma} d^2z \sum_{l=1}^{\dim n_+} \left( c^l D_{\bar{z}} b^l + \bar{c}^l D_z \bar{b}^l \right). \tag{3.20}
 \end{aligned}$$

From the equations of motion by varying the  $\tilde{B}^l$ 's, we have the conditions  $\tilde{A}_{\bar{z}}^l = 0$  for  $l = 1, \dots, \dim n_+$ . From the equations of motion by varying the  $\tilde{\bar{B}}^l$ 's, we also have the conditions  $\tilde{A}_z^l = 0$  for  $l = 1, \dots, \dim n_+$ . Thus, the partition function of the  $B$ -gauged WZW model can also be expressed as

$$Z_G = \int [g^{-1} dg, db, dc, d\bar{b}, d\bar{c}] \exp \left( i S_{\text{WZW}}(g) + \frac{ik'}{2\pi} \int_{\Sigma} d^2z \text{Tr}(c \cdot \partial_{\bar{z}} b)(z) + \text{Tr}(\bar{c} \cdot \partial_z \bar{b})(\bar{z}) \right), \tag{3.21}$$

where the *holomorphic* BRST variations of the fields which leave the effective action in (3.21) *invariant* are now given by

$$\begin{aligned}
 \delta_{\text{BRST}}(g) &= -c^m t_m^+ g, & \delta_{\text{BRST}}(c^l) &= -\frac{1}{2} f_{mk}^l c^m c^k, & \delta_{\text{BRST}}(b^l) &= J_+^l + M_+^l - f_{mk}^l b^m c^k, \\
 \delta_{\text{BRST}}(\text{others}) &= 0. \tag{3.22}
 \end{aligned}$$

The holomorphic BRST-charge generating the field variations in (3.22) will be given by

$$Q_{\text{BRST}} = \oint \frac{dz}{2\pi i} \left( \sum_{l=1}^{\dim n_+} c^l(z) (J_+^l(z) + M_+^l) - \frac{1}{2} \sum_{l=1}^{\dim n_+} f_{mk}^l b^m c^l c^k(z) \right). \tag{3.23}$$

The free-field action of the left-moving ghost fields in (3.21) implies that we have the usual OPE's of  $(\dim n_+)$  free  $bc$  systems. From these free  $bc$  OPE's, one can verify that  $Q_{\text{BRST}}$  in (3.23) will indeed generate the field variations in (3.22).

Though we did not make this obvious in our discussion above, by integrating out the  $\tilde{A}_{\bar{z}}^l$ 's in (3.14), and using the above conditions  $\tilde{A}_{\bar{z}}^l = 0$  for  $l = 1, \dots, \dim n_+$ , we find that we actually have the relations  $(J_+^l(z) + M_+^l) = 0$  for  $l = 1, \dots, \dim n_+$ . These relations — involving the current associated to the Borel subalgebra  $\mathfrak{b}$  of the group  $B$  that we are modding out from  $G$ —will lead us directly to the correct form of the holomorphic stress tensor for the gauged WZW model without reference to a coset formalism, as we shall see shortly.

Notice that physically consistent with the holomorphic chiral algebra of the purely bosonic sector of the twisted sigma-model on  $G/B$ , there are currents  $J^a(z)$  (given by the  $J_+^l(z)$ 's,  $J_-^l(z)$ 's and the  $J_c^l(z)$ 's) in the holomorphic BRST-cohomology of the non-dynamically  $B$ -gauged WZW model on  $G$ , where  $a = 1, 2, \dots, \dim \mathfrak{sl}_N$ , that generate an

affine  $G$  OPE-algebra at level  $k'$ . As such, one can construct a holomorphic stress tensor using the Sugawara formalism as

$$T_G(z) = \frac{:d_{ab}(J^a J^b)(z):}{k' + h^\vee}. \tag{3.24}$$

However, as shown above, one will have the conditions  $J_+^l = -M_+^l$  for  $l = 1, 2, \dots, \dim \mathfrak{n}_+$ . In order that the conformal dimensions of the  $J_+^l$ 's be compatible with these conditions, one must define a modified holomorphic stress tensor:

$$T_{\text{modified}}(z) = T_G(z) + \vec{l} \cdot \partial \vec{J}_c(z), \tag{3.25}$$

where  $\vec{J}_c(z)$  is a  $\text{rank}(\mathfrak{g})$ -dimensional vector with components being the  $J_c^l$  currents associated to the Cartan subalgebra  $\mathfrak{c}$ , and  $\vec{l}$  is a sum of simple, positive roots of  $\mathfrak{g}$ . In order for the above conditions on the  $J_+^l$ 's to be compatible with the fact that  $Q_{\text{BRST}}$  generating the holomorphic variations of the fields must be a scalar of dimension zero, the  $(\dim \mathfrak{n}_+)$ -set of left-moving ghost systems  $(b^l, c^l)$  must have conformal dimensions  $(h^l, 1 - h^l)$  for  $l = 1, 2, \dots, \dim \mathfrak{n}_+$ , where  $h^l$  is the conformal dimension of the corresponding  $J_+^l$  current under  $T_{\text{modified}}(z)$ . With all these points in mind, and by including the holomorphic stress tensor contribution from the action of the free left-moving ghost fields, we can write the total holomorphic stress tensor of the  $B$ -gauged WZW model on  $G$  as

$$T_{B\text{-gauged}}(z) = \frac{:d_{ab}(J^a J^b)(z):}{k' + h^\vee} + \sum_{a=1}^{\dim \mathfrak{c}} \partial_z J_c^a(z) - \sum_{l \in \Delta_+} \left[ h^l b^l \partial_z c^l(z) + (h^l - 1)(\partial_z b^l c^l)(z) \right], \tag{3.26}$$

where  $\Delta_+$  is the set of positive roots of  $\mathfrak{g}$ , and  $\rho^\vee$  is the ‘‘dual Weyl vector’’ of  $\mathfrak{g}$ , such that for  $\alpha \in \Delta_+$ , we have  $(\rho^\vee, \alpha) = 1$  if and only if  $\alpha$  is a simple root of  $\mathfrak{g}$ .

### 3.3 The $B$ -gauged WZW model on $G$ and the $\mathcal{W}_{k'}(\widehat{\mathfrak{g}})$ algebra

We shall now show that as one would expect from its role as an equivalent description of the holomorphic chiral algebra of the purely bosonic sector of the twisted sigma-model on  $G/B$ , the holomorphic BRST-cohomology of the  $B$ -gauged WZW model on  $G$  will contain local operators whose Laurent modes generate a  $\mathcal{W}_{k'}(\widehat{\mathfrak{g}})$  algebra.

To this end, let us first review a purely algebraic approach to generating  $\mathcal{W}_{k'}(\widehat{\mathfrak{g}})$ , the  $\mathcal{W}$ -algebra associated to the affine algebra  $\widehat{\mathfrak{g}}$  at level  $k'$ . This approach is known as the quantum Drinfeld-Sokolov (DS) reduction scheme [8, 29].

In general, the quantum DS-reduction scheme can be summarised as the following steps. Firstly, one starts with a triple  $(\widehat{\mathfrak{g}}, \widehat{\mathfrak{g}}', \chi)$ , where  $\widehat{\mathfrak{g}}'$  is an affine subalgebra of  $\widehat{\mathfrak{g}}$  at level  $k'$ , and  $\chi$  is a 1-dimensional representation of  $\widehat{\mathfrak{g}}'$ . Next, one imposes the first class constraints  $g \sim \chi(g)$ ,  $\forall g \in \widehat{\mathfrak{g}}'$ , via a BRST procedure. The cohomology of the BRST operator  $Q$  on the set of normal-ordered expressions in currents, ghosts and their derivatives, is what is called the Hecke algebra  $H_Q^i(\widehat{\mathfrak{g}}, \widehat{\mathfrak{g}}', \chi)$  of the triple  $(\widehat{\mathfrak{g}}, \widehat{\mathfrak{g}}', \chi)$ . For generic values of  $k'$ , the Hecke algebra vanishes for  $i \neq 0$ , and the existing zeroth cohomology  $H_Q^0(\widehat{\mathfrak{g}}, \widehat{\mathfrak{g}}', \chi)$ , is just spanned by a set of local operators associated to the triple  $(\widehat{\mathfrak{g}}, \widehat{\mathfrak{g}}', \chi)$ , whose Laurent

modes generate a closed  $\mathcal{W}$ -algebra. We shall denote the  $\mathcal{W}$ -algebra associated with this set of operators as  $\mathcal{W}_{\text{DS}}[\widehat{\mathfrak{g}}, \widehat{\mathfrak{g}}', \chi]$ . Note that  $\mathcal{W}_{\text{DS}}[\widehat{\mathfrak{g}}, \widehat{\mathfrak{g}}', \chi]$  is just  $\mathcal{W}_{k'}(\widehat{\mathfrak{g}})$ . Let us be more explicit about how one can go about defining  $\mathcal{W}_{\text{DS}}[\widehat{\mathfrak{g}}, \widehat{\mathfrak{g}}', \chi]$  and therefore  $\mathcal{W}_{k'}(\widehat{\mathfrak{g}})$ , now that we have sketched the general idea behind the DS-reduction scheme.

In order for  $\mathcal{W}_{\text{DS}}[\widehat{\mathfrak{g}}, \widehat{\mathfrak{g}}', \chi]$  to be a  $\mathcal{W}$ -algebra, one has to suitably choose the triple  $(\widehat{\mathfrak{g}}, \widehat{\mathfrak{g}}', \chi)$ . A suitable triple can be obtained by considering a principal  $\mathfrak{sl}_2$  embedding in  $\mathfrak{g}$ . Let us now describe this embedding. Suppose we have an  $\mathfrak{sl}_2$  subalgebra  $\{t_3, t_+, t_-\}$  of  $\mathfrak{g}$ . The adjoint representation of  $\mathfrak{g}$  decomposes into  $\mathfrak{sl}_2$  representations of spin  $j_k$ ,  $k = 1, \dots, s$ , for example. Then, one may write the  $\widehat{\mathfrak{g}}$  current  $J(z) = \sum_a^{\dim \mathfrak{g}} J^a(z)t_a$  as

$$J(z) = \sum_{k=1}^s \sum_{m=-j_k}^{j_k} J^{k,m}(z)t_{k,m} \tag{3.27}$$

where  $t_{k,m}$  corresponds to the generator of spin  $j_k$  and isospin  $m$  under the  $\mathfrak{sl}_2$  subalgebra. In particular, we have the correspondences  $t_{1,1} = t_+$ ,  $t_{1,0} = t_3$ , and  $t_{1,-1} = t_-$ . The  $\mathfrak{sl}_2$  subalgebra  $t_3, t_+, t_-$  can be characterized by a “dual Weyl vector”  $\rho^\vee$ , i.e., as mentioned above, for  $\alpha \in \Delta_+$ , where  $\Delta_+$  is the set of positive roots of  $\mathfrak{g}$ , we have  $(\rho^\vee, \alpha) = 1$  if and only if  $\alpha$  is a simple root of  $\mathfrak{g}$ . The  $\mathfrak{sl}_2$  root  $\hat{\alpha}$  is given by  $\hat{\alpha} = \rho/(\rho, \rho)$ , and  $t_3 = \rho \cdot \mathbf{c}$ , where  $\mathbf{c}$  is the Cartan subalgebra of  $\mathfrak{g}$ .

Take  $\widehat{\mathfrak{g}}'$  to be the affine Lie subalgebra  $\widehat{\mathfrak{n}}_+$  generated by all  $J^{k,m}(z)$ ,  $m > 0$ . Denoting the currents corresponding to positive roots  $\alpha$  by  $J^\alpha(z)$ , and choosing  $t_{1,1} = \sum_i e^{\alpha_i}$ , one can then impose the condition (which realises the required first-class constraint  $g \sim \chi(g)$ )

$$\chi_{\text{DS}}(J^\alpha(z)) = 1 \text{ (for simple roots } \alpha_i), \quad \chi(J^\alpha(z)) = 0 \text{ (otherwise)}. \tag{3.28}$$

Next, we introduce pairs of ghost fields  $(b^\alpha(z), c_\alpha(z))$ , one for every positive root  $\alpha \in \Delta_+$ . By definition, they obey the OPE  $b_\alpha(z)c_\beta(z') \sim \delta_{\alpha\beta}/(z - z')$ , where the  $\alpha, \beta$  (and  $\gamma$ ) indices run over the basis of  $\mathfrak{n}_+$ . The BRST operator that is consistent with (3.28) will then be given by  $Q = Q_0 + Q_1$ , where

$$Q_0 = \oint \frac{dz}{2\pi i} \left( J^\alpha(z)c_\alpha(z) - \frac{1}{2}f_\gamma^{\alpha\beta}(b^\gamma c_\alpha c_\beta)(z) \right) \tag{3.29}$$

is the standard differential associated to  $\widehat{\mathfrak{n}}_+$ ,  $f_\gamma^{\alpha\beta}$  are the structure constants of  $\mathfrak{n}_+$ , and

$$Q_1 = - \oint \frac{dz}{2\pi i} \chi_{\text{DS}}(J^\alpha(z))c_\alpha(z). \tag{3.30}$$

They satisfy

$$Q^2 = Q_0^2 = Q_1^2 = \{Q_0, Q_1\} = 0. \tag{3.31}$$

The resulting  $Q$ -cohomology is just the Hecke algebra  $H_Q^0(\widehat{\mathfrak{g}}, \widehat{\mathfrak{g}}', \chi)$ , which is spanned by a set of local operators whose Laurent modes generate  $\mathcal{W}_{\text{DS}}[\widehat{\mathfrak{g}}, \widehat{\mathfrak{g}}', \chi] = \mathcal{W}_{k'}(\widehat{\mathfrak{g}})$ . Note that (3.31) implies that one can compute the Hecke algebra via a spectral sequence of a double complex with differentials being  $Q_0$  and  $Q_1$  accordingly — this strategy has indeed been employed in [30] to compute explicitly the generators of the  $\mathcal{W}_2 = \mathcal{W}_{k'}(\widehat{\mathfrak{sl}}_2)$  and  $\mathcal{W}_3 = \mathcal{W}_{k'}(\widehat{\mathfrak{sl}}_3)$  OPE

algebras with central charges  $c = 13 - 6(k' + 2) - 6/(k' + 2)$  and  $c = 50 - 24(k' + 3) - 24/(k' + 3)$  respectively.

The variation of the various fields under the action of  $Q$  can also be computed using the OPE's of the affine algebra  $\widehat{\mathfrak{g}}$ , the OPE's of the ghost fields, and the explicit forms of  $Q_0$  and  $Q_1$  in (3.29) and (3.30) above, and they are given by

$$\delta c_\alpha(z) = -\frac{1}{2} f_\alpha^{\beta\gamma} (c_\beta c_\gamma)(z), \tag{3.32}$$

$$\delta b^\alpha(z) = J^\alpha(z) - \chi_{\text{DS}}(J^\alpha(z)) - f_\gamma^{\alpha\beta} (b^\gamma c_\beta)(z). \tag{3.33}$$

Note also that  $\mathcal{W}_{\text{DS}}[\widehat{\mathfrak{g}}, \widehat{\mathfrak{g}}', \chi]$  and thus  $\mathcal{W}_{k'}(\widehat{\mathfrak{g}})$ , will at least contain the Virasoro algebra. The explicit form of the stress tensor whose Laurent modes will generate the Virasoro algebra is (after omitting the normal-ordering symbol)

$$T_{\text{DS}}(z) = \frac{d_{ab} J^a(z) J^b(z)}{(k' + h^\vee)} + \sum_{c=1}^{\dim \mathfrak{c}} \partial_z J^c(z) + \sum_{\alpha \in \Delta_+} ((\rho^\vee, \alpha) - 1) b^\alpha \partial_z c_\alpha(z) + (\rho^\vee, \alpha) (\partial_z b^\alpha c_\alpha)(z), \tag{3.34}$$

where the  $J^c(z)$ 's are just the affine currents that are valued in the Cartan subalgebra  $\mathfrak{c}$  of the Lie algebra  $\mathfrak{g}$ . Note that with respect to  $T_{\text{DS}}(z)$ , the conformal dimensions of the pair  $(b^\alpha(z), c_\alpha(z))$  will be given by  $(1 - (\rho^\vee, \alpha), (\rho^\vee, \alpha))$ . The central charge of this Virasoro subalgebra and therefore that of  $\mathcal{W}_{k'}(\widehat{\mathfrak{g}})$ , will be given by

$$c(k') = \frac{k' \dim \mathfrak{g}}{(k' + h^\vee)} - 12k' |\rho^\vee|^2 - 2 \sum_{\alpha \in \Delta_+} (6(\rho^\vee, \alpha)^2 - 6(\rho^\vee, \alpha) + 1). \tag{3.35}$$

Notice at this point about the  $B$ -gauged WZW model on  $B$ , that for any  $J_+^l$  with  $h^l \neq 0$ , the corresponding  $M_+^l$  constant must be set to zero for consistency. This means from our above discussion, that one can identify  $M_+^l$  with  $-\chi_{\text{DS}}(J_+^l(z))$ . With this identification, one can see that the field variations in (3.22) agree with the field variations in (3.32) and (3.33). In addition, we find that  $Q_{\text{BRST}}$  in (3.23) also coincides with  $Q = Q_0 + Q_1$ , where  $Q_0$  and  $Q_1$  are given in (3.29) and (3.30), respectively. Moreover,  $T_{B\text{-gauged}}(z)$  of (3.26) is just  $T_{\text{DS}}(z)$  of (3.34). Hence, we see that the holomorphic BRST-cohomology of the  $B$ -gauged WZW model on  $G$  physically realises, in all generality, the purely algebraic DS-reduction scheme of generating the Hecke algebra.

We can summarise the results in this section as follows. Let us label the local operators of the Hecke algebra as  $T_{B\text{-gauged}}^{(s_i)}(z)$ , where  $i = 1, 2, \dots, \text{rank}(\mathfrak{g})$ ,  $s_i = e_i + 1$ ; the  $e_i$ 's being the exponents of  $\mathfrak{g}$ , and  $T_{B\text{-gauged}}^{(s_i)}(z)$  are higher spin- $s_i$  analogs of  $T_{B\text{-gauged}}(z)$ , where  $T_{B\text{-gauged}}^{(2)}(z) = T_{B\text{-gauged}}(z)$ . Then, we find that the holomorphic BRST-cohomology of the  $B$ -gauged WZW model on  $G$ , will be spanned by local operators  $T_{B\text{-gauged}}^{(s_i)}(z)$  whose Laurent modes will generate a  $\mathcal{W}_{k'}(\widehat{\mathfrak{g}})$  algebra with central charge (3.35). Consequently, the classical limit of  $\mathcal{W}_{k'}(\widehat{\mathfrak{g}})$ , i.e.,  $\mathcal{W}_\infty(\widehat{\mathfrak{g}})$ , will be given by the Poisson  $\mathcal{W}$ -algebra generated by the Laurent modes of the classical fields which lie in the classical, holomorphic BRST-cohomology of the  $B$ -gauged WZW model on  $G$ , that are the *classical* counterparts of the local operators  $T_{B\text{-gauged}}^{(s_i)}(z)$ . We shall discuss this set of classical fields next, and their role

in an isomorphism of classical  $\mathcal{W}$ -algebras and a level relation which underlie a geometric Langlands correspondence for  $G$ .

#### 4. A geometric Langlands correspondence for $G$

In this section, we will use what we have learnt in section 2 and section 3 about  $\mathfrak{z}(\widehat{\mathfrak{g}})$ ,  $\mathcal{W}_{k'}(\widehat{\mathfrak{g}})$  and the dual description afforded by the  $B$ -gauged WZW model on  $G$ , to show that an equivalence — at the level of the holomorphic chiral algebra — between a bosonic string on  $G/B$  and a  $B$ -gauged version of itself on  $G$ , will necessarily imply an isomorphism  $\mathfrak{z}(\widehat{\mathfrak{g}}) \simeq \mathcal{W}_\infty(L\widehat{\mathfrak{g}})$  of classical  $\mathcal{W}$ -algebras and the relation  $(k + h^\vee)r^\vee = (k' + Lh^\vee)^{-1}$  which underlie a geometric Langlands correspondence for  $G$ .

##### 4.1 The corresponding classical chiral algebra of the $B$ -gauged WZW model on $G$

Let us start by determining the observables of the  $B$ -gauged WZW model on  $G$  which correspond to the  $S^{(s_i)}(z)$  fields of the  $\psi^{\bar{j}}$ -independent, purely bosonic sector of the twisted sigma-model on  $X = G/B$ . Firstly, since the  $S^{(s_i)}(z)$ 's lie in the classical, holomorphic chiral algebra, the corresponding observables must also lie in the *classical*, holomorphic BRST-cohomology (or chiral algebra) of the  $B$ -gauged WZW model on  $G$ . Secondly, an observable corresponding to  $S^{(s_i)}(z)$  must also have spin  $s_i$ , and moreover, it must also generate the same (classical) symmetry in the gauged WZW model as that generated by  $S^{(s_i)}(z)$  in the sigma-model.

Now, recall that the *quantum* definition of the  $S^{(s_i)}(z)$ 's at  $k \neq -h^\vee$ , is given by  $S^{(s_i)}(z) = (k + h^\vee)T^{(s_i)}(z)$ . Since the  $S^{(s_i)}(z)$ 's cease to exist as quantum operators at  $k = -h^\vee$ , this must also be true of the corresponding observables in the gauged WZW model. Recall also that at  $k \neq -h^\vee$ , the (Laurent modes of the)  $S^{(s_i)}(z)$ 's generate a  $\mathcal{W}_k(\widehat{\mathfrak{g}})$  algebra. Similarly, the (Laurent modes of the)  $T_{B\text{-gauged}}^{(s_i)}(z)$ 's in the holomorphic chiral algebra of the gauged WZW model — each having spin  $s_i$ —generate a  $\mathcal{W}_{k'}(\widehat{\mathfrak{g}})$  algebra too. A little thought will then reveal that one can consistently identify  $S^{(s_i)}(z)$  with  $\overline{T}_{\text{classical}}^{(s_i)}(z)$ —the classical field counterpart of the local operator  $\overline{T}_{B\text{-gauged}}^{(s_i)}(z) = (k + h^\vee)T_{B\text{-gauged}}^{(s_i)}(z)$ . One can see this as follows. Firstly, notice that as required,  $\overline{T}_{\text{classical}}^{(s_i)}(z)$  is a spin- $s_i$  field that lies in the classical, holomorphic chiral algebra of the gauged WZW model at  $k = -h^\vee$ — at  $k = -h^\vee$ ,  $\overline{T}_{B\text{-gauged}}^{(s_i)}(z)$  will act by zero in its OPE's with any other operator, i.e., it will cease to exist as a quantum operator, and will reduce to a purely classical observable  $\overline{T}_{\text{classical}}^{(s_i)}(z)$ . Secondly, since the shift in  $h^\vee$  in the factor  $(k + h^\vee)$  is due to a quantum renormalisation effect as explained earlier, it will mean that  $\overline{T}_{\text{classical}}^{(s_i)}(z) = -h^\vee \cdot T_{\text{classical}}^{(s_i)}(z)$  at  $k = -h^\vee$ , where  $T_{\text{classical}}^{(s_i)}(z)$  is the classical counterpart of  $T_{B\text{-gauged}}^{(s_i)}(z)$ . This means that the  $\overline{T}_{\text{classical}}^{(s_i)}(z)$ 's will generate the same classical  $\mathcal{W}$ -symmetries in the gauged WZW model as those generated by the  $S^{(s_i)}(z)$ 's in the sigma-model.

In summary, one can identify the local  $S^{(s_i)}(z)$  fields in the classical, holomorphic chiral algebra of the  $\psi^{\bar{j}}$ -independent, purely bosonic sector of the twisted sigma-model on

$X = G/B$ , with the local fields  $\overline{T}_{\text{classical}}^{(s_i)}(z)$  in the classical, holomorphic chiral algebra of the  $B$ -gauged WZW model on  $G$ .

## 4.2 An isomorphism of classical $\mathcal{W}$ -algebras and a geometric Langlands correspondence for $G$

We have seen how, from an equivalence — at the level of the holomorphic chiral algebra — between the purely bosonic sector of the twisted sigma-model on  $G/B$  and the  $B$ -gauged WZW model on  $G$ , one can identify the  $S^{(s_i)}(z)$ 's with the  $\overline{T}_{\text{classical}}^{(s_i)}(z)$ 's. This identification will in turn imply that the Laurent modes of the local  $S^{(s_i)}(z)$  and  $\overline{T}_{\text{classical}}^{(s_i)}(z)$  fields ought to generate the same classical  $\mathcal{W}$ -algebra with identical central charges.

What is the central charge of the classical  $\mathcal{W}$ -algebra generated by the Laurent modes of the  $\overline{T}_{\text{classical}}^{(s_i)}(z)$  fields? To ascertain this, first note that the central charge of any (classical)  $\mathcal{W}$ -algebra will be given by the central charge of its (classical) Virasoro subalgebra. Next, note that the the Virasoro modes  $\hat{L}_n^{(2)}$  of  $T_{B\text{-gauged}}^{(2)}(z) = \sum_n \hat{L}_n^{(2)} z^{-n-2}$ , will obey the following commutator relation

$$[\hat{L}_n^{(2)}, \hat{L}_m^{(2)}] = (n - m) \hat{L}_{n+m}^{(2)} + \frac{c(k')}{12} (n^3 - n) \delta_{n,-m} \quad (4.1)$$

at the quantum level, where  $c(k')$  is given in (3.35). Therefore, the commutator relations involving the  $\hat{L}_n^{(2)}$  Virasoro modes of  $\overline{T}_{B\text{-gauged}}^{(2)}(z) = \sum_n \hat{L}_n^{(2)} z^{-n-2}$ , will be given by

$$[\hat{L}_n^{(2)}, \hat{L}_m^{(2)}] = (k + h^\vee) \left[ (n - m) \hat{L}_{n+m}^{(2)} + \frac{c(k', k)}{12} (n^3 - n) \delta_{n,-m} \right], \quad (4.2)$$

where  $c(k', k) = c(k')(k + h^\vee)$ . At  $k = -h^\vee$ ,  $\overline{T}_{B\text{-gauged}}^{(2)}(z)$  will cease to have a quantum definition, and it will reduce to its classical field counterpart  $\overline{T}_{\text{classical}}^{(2)}(z)$ . Consequently, the  $k \rightarrow -h^\vee$  (and  $k' \rightarrow \infty$ ) limit of the commutator relation in (4.2), can be interpreted as its classical limit. Therefore, one can view the term  $(k + h^\vee)$  in (4.2) as the parameter  $i\hbar$ , where  $\hbar \rightarrow 0$  is equivalent to the classical limit of the commutator relations. Since in a quantisation procedure, we go from  $\{\overline{L}_n^{(2)}, \overline{L}_m^{(2)}\}_{P.B.} \rightarrow \frac{1}{i\hbar} [\hat{L}_n^{(2)}, \hat{L}_m^{(2)}]$ , going in reverse would give us the classical Poisson bracket relation

$$\{\overline{L}_n^{(2)}, \overline{L}_m^{(2)}\}_{P.B.} = (n - m) \overline{L}_{n+m}^{(2)} + \frac{c(k', k)_{k \rightarrow -h^\vee, k' \rightarrow \infty}}{12} (n^3 - n) \delta_{n,-m}, \quad (4.3)$$

where  $\overline{T}_{\text{classical}}^{(2)}(z) = \sum_n \overline{L}_n^{(2)} z^{-n-2}$ . Hence, the well-defined central charge of the classical  $\mathcal{W}_\infty(\widehat{\mathfrak{g}})$  algebra generated by the Laurent modes  $\overline{L}_m^{(s_i)}$  of the  $\overline{T}_{\text{classical}}^{(s_i)}(z)$  fields, will be given by  $c(k', k)_{k \rightarrow -h^\vee, k' \rightarrow \infty}$ .

On the other hand, recall from our earlier discussion that the Laurent modes of the  $S^{(s_i)}(z)$  fields will generate a classical  $\mathcal{W}_{-h^\vee}(\widehat{\mathfrak{g}})$  algebra that contains a classical Virasoro subalgebra of central charge  $c = -h^\vee \dim(\mathfrak{g})$  given by

$$\{S_n^{(2)}, S_m^{(2)}\}_{P.B.} = (n - m) S_{n+m}^{(2)} - \frac{h^\vee \dim(\mathfrak{g})}{12} (n^3 - n) \delta_{n,-m}. \quad (4.4)$$

Hence, the well-defined central charge of the classical  $\mathcal{W}_{-h^\vee}(\widehat{\mathfrak{g}})$  algebra generated by the Laurent modes  $S_m^{(s_i)}$  of the  $S^{(s_i)}(z)$  fields, will be given by  $c = -h^\vee \dim(\mathfrak{g})$ . Therefore, since the classical  $\mathcal{W}$ -algebras generated by the  $S_n^{(s_i)}$ 's and the  $\overline{L}_n^{(s_i)}$ 's ought to be isomorphic with the same central charge, it means that we must have the relation

$$c(k, k')_{k \rightarrow -h^\vee, k' \rightarrow \infty} = -h^\vee \dim(\mathfrak{g}). \tag{4.5}$$

Note at this point that one can rewrite  $c(k')$  as [17]

$$c(k') = l - 12|\alpha_+ \rho + \alpha_- \rho^\vee|^2, \tag{4.6}$$

where  $l = \text{rank}(\mathfrak{g})$ ,  $\alpha_+ \alpha_- = 1$ ,  $\alpha_- = -\sqrt{k' + h^\vee}$ , and  $\rho$  is the Weyl vector. Since in our case of a simply-laced Lie algebra  $\mathfrak{g}$ , we have  $\mathfrak{g} = {}^L\mathfrak{g}$ , it will also mean that we have  $\rho = \rho^\vee$ . One can then simplify  $c(k')$  to

$$c(k') = l - 12|\rho|^2 \left( k' + h^\vee + \frac{1}{k' + h^\vee} - 2 \right). \tag{4.7}$$

From the Freudenthal-de Vries strange formula [31]

$$\frac{|\rho|^2}{2h^\vee} = \frac{\dim(\mathfrak{g})}{24}, \tag{4.8}$$

and the expression for  $c(k')$  in (4.7), we find that

$$c(k', k) = (k + h^\vee) \left[ l + 2h^\vee \dim(\mathfrak{g}) - h^\vee \dim(\mathfrak{g}) \left( k' + h^\vee + \frac{1}{k' + h^\vee} \right) \right]. \tag{4.9}$$

In the limit  $k \rightarrow -h^\vee$  and  $k' \rightarrow \infty$ , we find that

$$c(k', k)_{k \rightarrow -h^\vee, k' \rightarrow \infty} = -h^\vee \dim(\mathfrak{g}) \tag{4.10}$$

if and only if

$$(k + h^\vee)(k' + h^\vee) = 1. \tag{4.11}$$

Finally, recall that  $\mathcal{W}_{-h^\vee}(\widehat{\mathfrak{g}}) \simeq \mathfrak{z}(\widehat{\mathfrak{g}})$ , and since  $\mathfrak{g} = {}^L\mathfrak{g}$  for  $\mathfrak{g}$  simply-laced, we will also have  $h^\vee = {}^L h^\vee$  and  $r^\vee = 1$ . Hence, we see that an equivalence — at the level of the holomorphic chiral algebra — between the purely bosonic,  $\psi^{\bar{j}}$ -independent sector of the twisted sigma-model on  $G/B$  and the  $B$ -gauged WZW model on  $G$ , would imply an isomorphism of Poisson algebras

$$\mathfrak{z}(\widehat{\mathfrak{g}}) \simeq \mathcal{W}_\infty({}^L\widehat{\mathfrak{g}}), \tag{4.12}$$

and the level relation

$$(k + h^\vee)r^\vee = \frac{1}{(k' + {}^L h^\vee)}. \tag{4.13}$$

Recall at this point that the purely bosonic,  $\psi^{\bar{j}}$ -independent sector of the twisted sigma-model on  $G/B$ , can be described, via (3.2), by a bosonic string on  $G/B$ . On the other hand, note that since a bosonic string on a group manifold  $G$  can be described as a WZW model on  $G$ , it will mean that the  $B$ -gauged WZW model on  $G$  can be interpreted as a  $B$ -gauged



bosonic string on  $G$ . Thus, we see that an equivalence, at the level of the holomorphic chiral algebra, between a bosonic string on  $G/B$  and a  $B$ -gauged version of itself on  $G$ —which, can be viewed as a consequence of the ubiquitous notion that one can always physically interpret a geometrical symmetry of the target space as a gauge symmetry in the worldsheet theory — will imply an isomorphism of classical  $\mathcal{W}$ -algebras and a level relation that underlie a geometric Langlands correspondence for any complex,  $ADE$ -group  $G$ ! Note that the correspondence between the  $k \rightarrow -h^\vee$  and  $k' \rightarrow \infty$  limits (within the context of the above Poisson algebras) is indeed consistent with the relation (4.13). These limits define a “classical” geometric Langlands correspondence. A “quantum” generalisation of the  $G$  correspondence can be defined for other values of  $k$  and  $k'$  that satisfy the relation (4.13), but with the isomorphism of (4.12) replaced by an isomorphism of *quantum*  $\mathcal{W}$ -algebras (derived from a DS-reduction scheme) associated to  $\widehat{\mathfrak{g}}$  at levels  $k$  and  $k'$  respectively [8].

## 5. The Hecke eigensheaves and Hecke operators

We shall now show, via the isomorphism  $\mathfrak{z}(\widehat{\mathfrak{g}}) \simeq \mathcal{W}_\infty(L\widehat{\mathfrak{g}})$  of classical  $\mathcal{W}$ -algebras demonstrated in section 4 above, how one can derive a correspondence between flat holomorphic  ${}^L G$ -bundles on the worldsheet  $\Sigma$  and Hecke eigensheaves on the moduli space  $\text{Bun}_G$  of holomorphic  $G$ -bundles on  $\Sigma$ , where  $\Sigma$  is a closed Riemann surface of any genus.<sup>8</sup> In the process, we shall be able to physically interpret the Hecke eigensheaves and Hecke operators in terms of the correlation functions of purely bosonic local operators in the holomorphic chiral algebra of the twisted  $(0, 2)$  sigma-model on the complex flag manifold  $G/B$ .

### 5.1 Hecke eigensheaves on $\text{Bun}_G$ and flat ${}^L G$ -bundles on $\Sigma$

**Local primary field operators.** As we will explain shortly, the correlation functions of local primary field operators can be associated to the sought-after Hecke eigensheaves. As such, let us begin by describing these operators in the twisted  $(0, 2)$  sigma-model on a complex flag manifold  $X = G/B$ . By definition, the holomorphic primary field operators  $\Phi_s^\lambda(z)$  of any theory with an affine  $G$  OPE-algebra obey [31]

$$J^a(z)\Phi_r^\lambda(z') \sim - \sum_s \frac{(t_\lambda^a)_{rs} \Phi_s^\lambda(z')}{z - z'}, \tag{5.1}$$

where  $t_\lambda^a$  is a matrix in the  $\lambda$  representation of  $\mathfrak{sl}_N$ ,  $r, s = 1, \dots, \dim|\lambda|$ , and  $a = 1, \dots, \dim(\mathfrak{g})$ .

Since the  $\Phi_s^\lambda(z)$ 's obey OPE relations with the quantum operators  $J^a(z)$ , it will mean that they, like the  $J^a(z)$ 's, must exist as *quantum* bosonic operators of the sigma-model on  $X$ . And moreover, since (5.1) and the affine  $G$  OPE-algebra at the critical level generated by the  $J^a(z)$ 's in the  $\overline{Q}_+$ -cohomology of the quantum sigma-model together form a closed

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<sup>8</sup>Note that the twisted sigma-model on  $X$  has an anomaly given by  $c_1(X)c_1(\Sigma)$ . Hence, since  $c_1(X) \neq 0$  for any complex flag manifold  $X$ , the model is anomalous unless  $c_1(\Sigma) = 0$ . However, since we are only working locally on  $\Sigma$  via a local coordinate  $z$ , i.e., our arguments do not make any reference to the global geometry of the worldsheet which might contribute to a non-zero value of  $c_1(\Sigma)$ , we can ignore this anomaly. Thus, we are free to work with the sigma-model on *any*  $\Sigma$ .

OPE algebra, it will mean that the  $\Phi_s^\lambda(z)$ 's are also local operators in the  $\overline{Q}_+$ -cohomology of the sigma-model on  $X$  at the quantum level. From our  $\overline{Q}_+$ -Cech cohomology dictionary (as explained in appendix A of [1]), this means that the  $\Phi_s^\lambda(z)$ 's will correspond to classes in  $H^0(X, \mathcal{O}_X^{\text{ch}})$ , i.e., the global sections of the sheaf  $\mathcal{O}_X^{\text{ch}}$  of CDO's on  $X$ . Note that this observation is also consistent with (5.1), since one can generate other global sections of the sheaf  $\mathcal{O}_X^{\text{ch}}$  from the OPE's of existing global sections.

The fact that these operators can be described by global sections of the sheaf of CDO's on  $X$  means that they reside within the purely bosonic sector of the holomorphic chiral algebra of the underlying sigma-model on  $X$ . As we shall see, this observation will serve as a platform for a physical interpretation of the Hecke eigensheaves.

**Space of coinvariants.** Associated to the correlation functions of the above-described local primary field operators, is the concept of a space of coinvariants, which, in its interpretation as a sheaf over the moduli space of holomorphic  $G$ -bundles on  $\Sigma$  that we will clarify below, is directly related to the Hecke eigensheaves that we are looking for. Hence, let us now turn our attention to describing this space of coinvariants.

Notice that if the twisted sigma-model were to be conformal, i.e.,  $[\overline{Q}_+, T(z)] = 0$  even after quantum corrections, we would have a CFT operator-state isomorphism, such that any primary field operator  $\Phi_s^\lambda(z)$  would correspond to a state  $|\Phi_s^\lambda\rangle$  in the highest-weight representation of  $\widehat{\mathfrak{g}}$ . However, since the twisted sigma-model on a complex flag manifold  $G/B$  lacks a holomorphic stress tensor and is thus non-conformal, a  $\Phi_s^\lambda(z)$  operator will not have a one-to-one correspondence with a state  $|\Phi_s^\lambda\rangle$ . Rather, the states just furnish a module of the chiral algebra spanned by the local operators themselves.

Nevertheless, in the axiomatic CFT framework of a theory with an affine algebra  $\widehat{\mathfrak{g}}$ , the operator-state isomorphism is an axiom that is defined at the outset, and therefore, any primary field operator will be axiomatically associated to a state in the highest-weight representation of  $\widehat{\mathfrak{g}}$ . Bearing this in mind, now consider a general correlation function of  $n$  primary field operators such as  $\langle \Phi_s^{\lambda_1}(z_1) \dots \Phi_s^{\lambda_n}(z_n) \rangle$ . Note that it can be viewed, in the axiomatic CFT sense, as a map from a tensor product of  $n$  highest-weight representations of  $\widehat{\mathfrak{g}}$  to a complex number. Next, consider a variation of the correlation function under a *global*  $G$ -transformation, i.e.,  $\delta_\omega \langle \Phi_s^{\lambda_1}(z_1) \dots \Phi_s^{\lambda_n}(z_n) \rangle = \oint_C dz \sum_a \omega^a \langle J^a(z) \Phi_s^{\lambda_1}(z_1) \dots \Phi_s^{\lambda_n}(z_n) \rangle$ , where  $\omega^a$  is a *position-independent* scalar transformation parameter, and where  $C$  is a contour that encircles all the points  $z_1, \dots, z_n$  on  $\Sigma$ . Since all the  $J^a(z)$ 's are dimension-one conserved currents in the  $\overline{Q}_+$ -cohomology of the twisted sigma-model on  $G/B$ , they will generate a symmetry of the theory. In other words, we will have  $\delta_\omega \langle \Phi_s^{\lambda_1}(z_1) \dots \Phi_s^{\lambda_n}(z_n) \rangle = 0$ , which is simply a statement of the global  $G$ -invariance of any theory with an affine  $G$  algebra. This last statement, together with the one preceding it, means that a general correlation function of  $n$  primary field operators  $\langle \Phi_s^{\lambda_1}(z_1) \dots \Phi_s^{\lambda_n}(z_n) \rangle$  will define a ‘‘conformal block’’ in the axiomatic CFT sense [8]. Proceeding from this mathematical definition of a ‘‘conformal block’’, the collection of operators  $\Phi_s^{\lambda_1}(z_1) \dots \Phi_s^{\lambda_n}(z_n)$  will define a vector  $\Phi$  in the dual space of coinvariants  $H_{\mathfrak{g}}(\Phi_s^{\lambda_1}(z_1) \dots \Phi_s^{\lambda_n}(z_n))$ , whereby the ‘‘conformal block’’ or correlation function  $\langle \Phi_s^{\lambda_1}(z_1) \dots \Phi_s^{\lambda_n}(z_n) \rangle$  can be computed as the square  $|\Phi|^2$  of length of  $\Phi$  with respect to a hermitian inner product on  $H_{\mathfrak{g}}(\Phi_s^{\lambda_1}(z_1) \dots \Phi_s^{\lambda_n}(z_n))$  [8]. All correlation

functions of primary field operators can be computed once this inner product is determined.

**Sheaf of coinvariants on  $\text{Bun}_G$ .** As mentioned above, what will be directly related to the Hecke eigensheaves is the sheaf of coinvariants on the moduli space  $\text{Bun}_G$  of holomorphic  $G$ -bundles on the worldsheet  $\Sigma$ . Let us now describe how this sheaf of coinvariants arises. However, before we proceed, let us first explain how holomorphic  $G$ -bundles on  $\Sigma$  can be consistently defined in the presence of an affine  $G$ -algebra in the sigma-model on  $X = G/B$ .

Recall that for the sigma-model on  $X = G/B$ , we have the OPE

$$J_a(z)J_b(w) \sim \frac{kd_{ab}}{(z-w)^2} + \sum_c f_{ab}{}^c \frac{J_c(w)}{(z-w)}. \tag{5.2}$$

where  $d_{ab}$  is the Cartan-Killing metric of  $\mathfrak{g}$ . Note also that since the above dimension-one current operators are holomorphic in  $\Sigma$ , they can be expanded in a Laurent expansion around the point  $w$  on  $\Sigma$  as

$$J_a(z) = \sum_n J_a^n(w)(z-w)^{-n-1}. \tag{5.3}$$

Consequently, from the above OPE, we will get the commuator relation

$$[J_a^n(w), J_b^m(w)] = \sum_c f_{ab}{}^c J_c^{n+m}(w) + (kd_{ab}) n \delta_{n+m,0}, \tag{5.4}$$

such that the Lie algebra  $\mathfrak{g}$  generated by the zero-modes of the currents will be given by

$$[J_a^0(w), J_b^0(w)] = \sum_c f_{ab}{}^c J_c^0(w). \tag{5.5}$$

One can then exponentiate the above generators that span  $\mathfrak{g}$  to define an element of  $G$ , and since these generators depend on the point  $w$  in  $\Sigma$ , it will mean that one can, via this exponential map, consistently define a non-trivial principal  $G$ -bundle on  $\Sigma$ . Moreover, this bundle will be holomorphic as the underlying generators only vary holomorphically in  $w$  on the worldsheet  $\Sigma$ .

Let us label the above-described holomorphic  $G$ -bundle on  $\Sigma$  as  $\mathcal{P}$ . Then, the space  $H_{\mathfrak{g}}(\Phi_s^{\lambda_1}(z_1) \dots \Phi_s^{\lambda_n}(z_n))$  of coinvariants will vary non-trivially under infinitesimal deformations of  $\mathcal{P}$ . As such, one can define a sheaf of coinvariants over the space  $\text{Bun}_G$  of all holomorphic  $G$ -bundles on  $\Sigma$ . Let us justify this statement next.

Firstly, note that with our description of  $\mathcal{P}$  via the affine  $G$ -algebra of the sigma-model on  $X$ , there is a mathematical theorem [9] which states that  $\text{Bun}_G$  is locally uniformized by the affine  $G$ -algebra. What this means is that the tangent space  $T_{\mathcal{P}}\text{Bun}_G$  to the point in  $\text{Bun}_G$  which corresponds to an  $G$ -bundle on  $\Sigma$  labelled by  $\mathcal{P}$ , will be isomorphic to the space  $H^1(\Sigma, \text{End}\mathcal{P})$  [9]. Moreover, deformations of  $\mathcal{P}$ , which correspond to displacements from this point in  $\text{Bun}_G$ , are generated by an element  $\eta(z) = J^a \eta_a(z)$  of the loop algebra of  $\mathfrak{g}$ , where  $\eta_a(z)$  is a *position-dependent* scalar deformation parameter (see section 17.1 of [9] and section 7.3 of [8]). With this in mind, let us again consider the  $n$ -point correlation function  $\langle \Phi_s^{\lambda_1}(z_1) \dots \Phi_s^{\lambda_n}(z_n) \rangle$ . By inserting  $\eta(z)$  into this correlation function, and computing the contour integral around the points  $z_1, \dots, z_n$ , we have

$\delta_\eta \langle \Phi_s^{\lambda_1}(z_1) \dots \Phi_s^{\lambda_n}(z_n) \rangle = \langle \oint_C dz \sum_a \eta_a(z) J^a(z) \Phi_s^{\lambda_1}(z_1) \dots \Phi_s^{\lambda_n}(z_n) \rangle$ , where  $C$  is a contour which encircles the points  $z_1, \dots, z_n$  on  $\Sigma$ , and  $\delta_\eta \langle \Phi_s^{\lambda_1}(z_1) \dots \Phi_s^{\lambda_n}(z_n) \rangle$  will be the variation of  $\langle \Phi_s^{\lambda_1}(z_1) \dots \Phi_s^{\lambda_n}(z_n) \rangle$  under an infinitesimal deformation of  $\mathcal{P}$  generated by  $\eta(z)$  (see eq. (7.9) of [8] and also [32]). Note that this variation does not vanish, since  $\eta_a(z)$ , unlike  $\omega$  earlier, is a position-dependent parameter of a *local*  $G$ -transformation. Therefore, as explained above, since the correlation function  $\langle \Phi_s^{\lambda_1}(z_1) \dots \Phi_s^{\lambda_n}(z_n) \rangle$  is associated to  $\Phi$  in the dual space of coinvariants  $H_{\mathfrak{g}}(\Phi_s^{\lambda_1}(z_1) \dots \Phi_s^{\lambda_n}(z_n))$ , one can see that  $\Phi$  must vary in  $H_{\mathfrak{g}}(\Phi_s^{\lambda_1}(z_1) \dots \Phi_s^{\lambda_n}(z_n))$  as one moves infinitesimally along a path in  $\text{Bun}_G$ . Since  $\Phi$  is just a vector in some basis of  $H_{\mathfrak{g}}(\Phi_s^{\lambda_1}(z_1) \dots \Phi_s^{\lambda_n}(z_n))$ , one could instead interpret  $\Phi$  to be fixed, while  $H_{\mathfrak{g}}(\Phi_s^{\lambda_1}(z_1) \dots \Phi_s^{\lambda_n}(z_n))$  varies as one moves infinitesimally along a path in  $\text{Bun}_G$ , as  $\mathcal{P}$  is subjected to infinitesimal deformations. Consequently, we have an interpretation of a *sheaf of coinvariants* on  $\text{Bun}_G$ , where the fibre of this sheaf over each point in  $\text{Bun}_G$  is just the space  $H_{\mathfrak{g}}(\Phi_s^{\lambda_1}(z_1) \dots \Phi_s^{\lambda_n}(z_n))$  of coinvariants corresponding to a particular bundle  $\mathcal{P}$  that one can consistently define over  $\Sigma$  using the affine  $G$ -algebra of the sigma-model on  $X = G/B$ . Note however, that since we are dealing with an affine  $G$ -algebra at the critical level  $k = -h^\vee$ , the dimension of the space of coinvariants will vary over different points in  $\text{Bun}_G$ . In other words, the sheaf of coinvariants on  $\text{Bun}_G$  does not have a structure of a vector bundle, since the fibre space of a vector bundle must have a fixed dimension over different points on the base. Put abstractly, this is because  $\widehat{\mathfrak{g}}$ -modules at the critical level may only be exponentiated to a subgroup of the Kac-Moody group  $\widehat{G}$ . Nevertheless, the sheaf of coinvariants is a twisted  $\mathcal{D}$ -module on  $\text{Bun}_G$  [8].

From the above discussion, one can also make the following physical observation. Notice that the variation  $\delta_\eta \langle \Phi_s^{\lambda_1}(z_1) \dots \Phi_s^{\lambda_n}(z_n) \rangle = \langle \oint_C dz \sum_a \eta_a(z) J^a(z) \Phi_s^{\lambda_1}(z_1) \dots \Phi_s^{\lambda_n}(z_n) \rangle$  in the correlation function as one moves along  $\text{Bun}_G$ , can be interpreted, at the lowest order in sigma-model perturbation theory, as a variation in the correlation function due to a *marginal* deformation of the sigma-model action by the term  $\oint dz \eta(z)$ . Since a deformation of the action by the dimensionless term  $\oint dz \eta(z)$  is tantamount to a displacement in the moduli space of the sigma-model itself, it will mean that  $\delta_\eta \langle \Phi_s^{\lambda_1}(z_1) \dots \Phi_s^{\lambda_n}(z_n) \rangle$  is also the change in the correlation function as one varies the moduli of the sigma-model. This implies that  $\text{Bun}_G$  will at least correspond to a subspace of the entire moduli space of the sigma-model on  $X = G/B$ . This should come as no surprise since  $\mathcal{P}$  is actually associated to the affine  $G$ -algebra of the sigma-model on  $X = G/B$  as explained, and moreover, the affine  $G$ -algebra does depend on the moduli of the sigma-model as mentioned in section 2.

Last but not least, note that the sheaf of coinvariants can also be obtained purely mathematically [8] via a localisation functor  $\Delta$ , which maps the chiral vertex algebra  $V_\chi$ —identifiable with all polynomials  $F(\mathcal{J}(z))$  (which exist in the chiral algebra of the twisted sigma-model on  $G/B$ ) that are defined over the field of complex numbers and the  $c$ -number operators  $S^{(s_i)}(z)$ , and that are of arbitrary positive degree in the quantum operator  $\mathcal{J}(z) = \frac{1}{(-n_1-1)! \dots (-n_m-1)!} : \partial_z^{-n_1-1} J^{a_1}(z) \dots \partial_z^{-n_m-1} J^{a_m}(z) :$ —to the corresponding sheaf  $\Delta(V_\chi)$  of coinvariants on  $\text{Bun}_G$ , where  $\chi$  denotes a parameterisation of  $V_\chi$  that depends on the choice of the set of  $S^{(s_i)}(z)$  fields for  $i = 1, \dots, \text{rank}(\mathfrak{g})$ . In other words, the sheaf of coinvariants

will be parameterised by  $\chi$ .<sup>9</sup> This observation is pivotal in the mathematical description of the correspondence between Hecke eigensheaves on  $\text{Bun}_G$  and flat holomorphic  ${}^L G$ -bundles on  $\Sigma$ , via the algebraic CFT approach to the geometric Langlands program [8]. As we will explain below, this parameterisation of the sheaf of coinvariants on  $\text{Bun}_G$  by the set of  $S^{s_i}(z)$  fields can be shown to arise physically in the sigma-model as well.

**A  $\mathfrak{z}(\widehat{\mathfrak{g}})$ -dependent realisation of the affine  $G$ -algebra at the critical level.** Before one can understand how, within the context of the sigma-model on  $X = G/B$ , the sheaf of coinvariants can be parameterised by a choice of the set of  $S^{s_i}(z)$  fields for  $i = 1, \dots, \text{rank}(\mathfrak{g})$ , it will be necessary for us to understand how one can achieve a  $\mathfrak{z}(\widehat{\mathfrak{g}})$ -dependent realisation of the affine  $G$  OPE algebra at  $k = -h^\vee$  spanned by the set of  $J^a(z)$  currents that correspond to classes in  $H^0(X, \mathcal{O}_X^{\text{ch}})$ .

To this end, let us first consider the set of local operators composed out of the  $n = \dim_{\mathbb{C}} X$  free  $\beta_i(z)$  and  $\gamma^i(z)$  fields of the  $n$  linear  $\beta\gamma$  systems associated to the sheaf of CDO's on  $X$ :

$$J_-^i(z) = \beta^{\alpha_i}(z) + \sum_{\varphi \in \Delta_+} : P_\varphi^i(\gamma^\alpha(z)) \beta^\varphi(z) :, \tag{5.6}$$

$$J_c^k(z) = - \sum_{\varphi \in \Delta_+} \varphi(h^k) : \gamma^\varphi(z) \beta^\varphi(z) :, \tag{5.7}$$

$$J_+^i(z) = \sum_{\varphi \in \Delta_+} : Q_\varphi^i(\gamma^\alpha(z)) \beta^\varphi(z) : + c_i \partial_z \gamma^{\alpha_i}(z), \tag{5.8}$$

where the subscripts  $\{\pm, c\}$  denote a Cartan decomposition of the Lie algebra  $\mathfrak{g}$  under which the  $J(z)$  local operators can be classified, the superscript  $\alpha_i$  denotes the free field that can be identified with the  $i^{\text{th}}$  positive root of  $\mathfrak{g}$  where  $i = 1, \dots, n$ ,  $h^k$  is an element of the Cartan subalgebra of  $\mathfrak{g}$  where  $k = 1, \dots, \text{rank}(\mathfrak{g})$ ,  $\varphi(h^k)$  is the  $k^{\text{th}}$  component of the root  $\varphi$ , the symbol  $\Delta_+$  denotes the set of positive roots of  $\mathfrak{g}$ , the  $c_i$ 's are complex constants, and lastly,  $P_\varphi^i, Q_\varphi^i$  are some polynomials in the  $\gamma^\alpha$  free fields.

Theorem 4.3 of [33] tells us that the Laurent modes of the above set of local operators  $\{J_\pm^i, J_c^k\}$  generate an affine  $G$ -algebra at the critical level  $k = -h^\vee$ , i.e., the set  $\{J_\pm^i, J_c^k\}$  will span an affine  $G$  OPE-algebra at the critical level  $k = -h^\vee$ . Moreover, the fact that the currents  $\{J_\pm^i, J_c^k\}$  are composed purely out of free  $\beta_i$  and  $\gamma^i$  fields, and the fact that there will always be classes in  $H^0(X, \mathcal{O}_X^{\text{ch}})$  which correspond to operators that generate an affine  $G$  OPE algebra [13], will together mean that the set of currents  $\{J_\pm^i, J_c^k\}$  must correspond (up to  $\overline{Q}_+$ -exact terms at worst) to classes in  $H^0(X, \mathcal{O}_X^{\text{ch}})$ . Equivalently, this means that the set of local current operators  $\{J_\pm^i, J_c^k\}$  will be  $\overline{Q}_+$ -closed and hence lie in the holomorphic chiral algebra of the twisted sigma-model on  $X = G/B$ .

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<sup>9</sup>Note that in order to be consistent with the notation used in the mathematical literature, we have chosen to use the symbol  $\chi$  to label the parameterisation of  $V_\chi$ . Hopefully,  $\chi$  that appears here and henceforth will not be confused with the one-dimensional representation  $\chi$  of  $\widehat{\mathfrak{g}}$  in section 3.

Next, let us consider a modification  $\{J_{\pm}^{i'}, J_c^{k'}\}$  of the set of currents  $\{J_{\pm}^i, J_c^k\}$ , where

$$J_-^{i'}(z) = \beta^{\alpha_i}(z) + \sum_{\varphi \in \Delta_+} : P_{\varphi}^i(\gamma^{\alpha}(z))\beta^{\varphi}(z) :, \quad (5.9)$$

$$J_c^{k'}(z) = - \sum_{\varphi \in \Delta_+} \varphi(h^k) : \gamma^{\varphi}(z)\beta^{\varphi}(z) : + b^i(z), \quad (5.10)$$

$$J_+^{i'}(z) = \sum_{\varphi \in \Delta_+} : Q_{\varphi}^i(\gamma^{\alpha}(z))\beta^{\varphi}(z) : + c_i \partial_z \gamma^{\alpha_i}(z) + b^i(z)\gamma^{\alpha_i}(z), \quad (5.11)$$

and the  $b^i(z)$ 's are just classical  $c$ -number functions that are holomorphic in  $z$  and of conformal dimension one — it can be Laurent expanded as  $b^i(z) = \sum_{n \in \mathbb{Z}} b_n^i z^{-n-1}$ .<sup>10</sup> Since the  $b^i(z)$ 's are classical fields, they will not participate as interacting (quantum) fields in any of the OPE's among the quantum operators  $\{J_+^{i'}, J_-^{i'}, J_3^{k'}\}$ . Rather, they will just act as a simple multiplication on the  $\gamma^{\alpha_i}(z)$  and  $\beta^{\alpha_i}(z)$  fields, or functions in them thereof. Moreover, this means that the  $b^i(z)$ 's must be trivial in the  $\overline{Q}_+$ -cohomology of the twisted sigma-model on  $G/B$  at the quantum level, i.e., it can be expressed as a  $\overline{Q}_+$ -exact term  $\{\overline{Q}_+, \dots\}$  in the *quantum* theory. Now, recall that we had the (non quantum-corrected) geometrical gluing relation  $\gamma^{\alpha_i} = g^{\alpha_i}(\gamma^{\alpha})$ , where each  $\gamma^{\alpha_i}$  and  $g^{\alpha_i}(\gamma^{\alpha})$  is defined in the open set  $U_1$  and  $U_2$  respectively of the intersection  $U_1 \cap U_2$  in  $X$ . This expression means that the  $\gamma^{\alpha_i}$ 's define global sections of the sheaf  $\widehat{\mathcal{O}}_X^{\text{ch}}$ . From our  $\overline{Q}_+$ -Cech cohomology dictionary, this will mean that each  $\gamma^{\alpha_i}(z)$  must correspond to an operator in the twisted sigma-model on  $X$  that is annihilated by  $\overline{Q}_+$  at the quantum level. This, together with the fact that  $b^i(z)$ 's can be expressed as  $\{\overline{Q}_+, \dots\}$ , will mean that the  $b^i(z)\gamma^{\alpha_i}(z)$  term in  $J_+^{i'}(z)$  of (5.11) above, can be written as a  $\overline{Q}_+$ -exact term  $\{\overline{Q}_+, \dots\}$ . Likewise, the  $b^i(z)$  term in  $J_c^{k'}(z)$  of (5.10) can also be written as a  $\overline{Q}_+$ -exact term  $\{\overline{Q}_+, \dots\}$ . Consequently, since  $\overline{Q}_+^2 = 0$  even at the quantum level,  $\{J_+^{i'}, J_-^{i'}, J_3^{i'}\}$  continues to be a set of quantum operators that are  $\overline{Q}_+$ -closed and non- $\overline{Q}_+$ -exact, i.e.,  $\{J_+^{i'}, J_-^{i'}, J_3^{i'}\}$  correspond to classes in  $H^0(X, \widehat{\mathcal{O}}_X^{\text{ch}})$ . Since the OPE's of  $\overline{Q}_+$ -exact terms such as  $b^i(z)\gamma^{\alpha_i}(z)$  and  $b^i(z)$  with the other  $\overline{Q}_+$ -closed terms such as  $(\sum_{\varphi \in \Delta_+} : Q_{\varphi}^i(\gamma^{\alpha})\beta^{\varphi} : + c_i \partial_z \gamma^{\alpha_i})$ ,  $(\beta^{\alpha_i} + \sum_{\varphi \in \Delta_+} : P_{\varphi}^i(\gamma^{\alpha})\beta^{\varphi} :)$ , and  $(-\sum_{\varphi \in \Delta_+} \varphi(h^k) : \gamma^{\varphi}\beta^{\varphi} :)$  that correspond respectively to the set of original operators  $J_+^i$ ,  $J_-^i$ , and  $J_c^k$ , must again result in  $\overline{Q}_+$ -exact terms that are trivial in  $\overline{Q}_+$ -cohomology, they can be discarded in the OPE's involving the set of operators  $\{J_+^{i'}, J_-^{i'}, J_3^{i'}\}$ , i.e., despite being expressed differently from the set of original operators  $\{J_+^i, J_-^i, J_c^k\}$ , the set of operators  $\{J_+^{i'}, J_-^{i'}, J_c^{k'}\}$  will persist to generate an affine  $G$  OPE-algebra at the critical level  $k = -h^{\vee}$ . In other words, via the set of modified operators  $\{J_{\pm}^{i'}, J_c^{k'}\}$  and their corresponding Laurent modes, we have a different realisation of the affine  $G$ -algebra at the critical level  $k = -h^{\vee}$ .

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<sup>10</sup>Note that the explicit expression of the  $b^i(z)$ 's cannot be arbitrary. It has to be chosen appropriately to ensure that the Segal-Sugawara tensor and its higher spin analogs given by the  $S^{(s_i)}(z)$ 's, can be identified with the space of  $L\mathfrak{g}$ -opers on the formal disc  $D$  in  $\Sigma$  as necessitated by the isomorphism  $\mathfrak{z}(\widehat{\mathfrak{g}}) \simeq \mathcal{W}_{\infty}(L\widehat{\mathfrak{g}})$  demonstrated earlier. For example, the expression of  $b(z)$  as  $\frac{1}{2}c(z)$  in the  $G = \text{SL}(2, \mathbb{C})$  case of [1] ensures that  $S'(z) = \frac{1}{4}c^2(z) - \frac{1}{2}\partial_z c(z)$  can be identified with a projective connection on  $D$  for each choice of  $c(z)$ . However, since the explicit form of the  $b^i(z)$ 's will not be required for our discussions, we shall not have anything more to say them.

This is consistent with Theorem 4.7 of [33], which states that the set  $\{J_{\pm}^{i'}, J_c^{k'}\}$  of modified operators will persist to generate an affine  $G$  OPE-algebra at the critical level  $k = -h^{\vee}$ .

Obviously, from (5.9)–(5.11), we see that the above realisation depends on the choice of the  $b^i(z)$ 's. What determines the  $b^i(z)$ 's then? To answer this, let us first recall that the Segal-Sugawara tensor  $S^{(2)'}(z)$  and its higher spin analogs  $S^{(s_i)'}(z)$  associated to the modified operators  $\{J_+^{i'}, J_-^{i'}, J_c^{k'}\} \in \{J^{a'}\}$ , can be expressed as  $S^{(s_i)'}(z) = \tilde{d}_{a_1 a_2 \dots a_{s_i}} : J^{a_1} J^{a_2} \dots J^{a_{s_i}}(z) :$  in the quantum theory. However, recall also that the original Segal-Sugawara tensor and its higher spin analogs, expressed as  $S^{(s_i)}(z) = \tilde{d}_{a_1 a_2 \dots a_{s_i}} : J^{a_1} J^{a_2} \dots J^{a_{s_i}}(z) :$  in terms of the original operators  $\{J_+^i, J_-^i, J_c^k\} \in \{J^a\}$ , act by zero in the quantum theory. This means that the non-vanishing contributions to any of the  $S^{(s_i)'}(z)$ 's come only from terms that involve the additional  $b^i(z)$  fields. In fact, it is true that the  $S^{(s_i)'}(z)$ 's also act by zero in the quantum theory at  $k = -h^{\vee}$ , since they are also defined via a Sugawara-type construction which results in their quantum definition being  $S^{(s_i)'}(z) = (k + h^{\vee})T^{(s_i)'}(z)$ . In other words, the  $S^{(s_i)'}(z)$ 's must be classical  $c$ -number fields of spin  $s_i$  that are holomorphic in  $z$ . This implies that the  $S^{(s_i)'}(z)$ 's will be expressed solely in terms of the  $c$ -number  $b^i(z)$  fields. An explicit example of this general statement has previously been discussed in the case of  $G = \text{SL}(2, \mathbf{C})$  in [1] — for  $G = \text{SL}(2, \mathbf{C})$ , we have the identification  $J_+^{i'} \leftrightarrow J'_+, J_-^{i'} \leftrightarrow J'_-, J_c^{k'} \leftrightarrow J'_3, S^{(2)'}(z) \leftrightarrow S'(z), b^i(z) \leftrightarrow \frac{1}{2}c(z)$  and  $S^{(2)'}(z) = \frac{1}{4}c^2(z) - \frac{1}{2}\partial_z c(z)$ , whereby the choice of  $S^{(2)'}(z)$  determines  $c(z)$ . Consequently, a choice of the set of  $S^{(s_i)'}(z)$  fields will determine the  $b^i(z)$  fields. Lastly, note that the  $S^{(s_i)'}(z)$  fields lie in the classical holomorphic chiral algebra of the purely bosonic sector of the twisted sigma-model on  $X = G/B$ , and their Laurent modes span the centre  $\mathfrak{z}(\widehat{\mathfrak{g}})$  of the completed universal enveloping algebra of  $\widehat{\mathfrak{g}}$  at the critical level  $k = -h^{\vee}$ . Hence, we effectively have a  $\mathfrak{z}(\widehat{\mathfrak{g}})$ -dependent realisation of the affine  $G$  (OPE) algebra at the critical level as claimed.

**A  $\mathfrak{z}(\widehat{\mathfrak{g}})$ -dependent parameterisation of the sheaf of coinvariants on  $\text{Bun}_G$ .** Now that we have seen how one can obtain a  $\mathfrak{z}(\widehat{\mathfrak{g}})$ -dependent realisation of the affine  $G$  (OPE) algebra at the critical level, we can proceed to explain how, within the context of the sigma-model on  $X = G/B$ , the sheaf of coinvariants on  $\text{Bun}_G$  can be parameterised by a choice of the fields  $S^{s_i}(z)$  for  $i = 1, \dots, \text{rank}(\mathfrak{g})$ .

To this end, notice that since the primary field operators  $\Phi_s^\lambda(z)$  are defined via the OPE's with the  $J^a(z)$  currents of the  $\widehat{\mathfrak{g}}$  algebra at the critical level in (5.1), a different realisation of the  $J^a(z)$  currents will also result in a different realisation of the  $\Phi_s^\lambda(z)$ 's. Consequently, we will have a  $\mathfrak{z}(\widehat{\mathfrak{g}})$ -dependent realisation of the primary field operators  $\Phi_s^\lambda(z)$ . This amounts to a  $\mathfrak{z}(\widehat{\mathfrak{g}})$ -dependent realisation of their  $n$ -point correlation functions  $\langle \Phi_s^{\lambda_1}(z_1) \dots \Phi_s^{\lambda_n}(z_n) \rangle$ . Since the correlation functions can be associated to a (vector in the) space of coinvariants as explained earlier, one will consequently have a  $\mathfrak{z}(\widehat{\mathfrak{g}})$ -dependent realisation of the sheaf of coinvariants on  $\text{Bun}_G$  as well, i.e., the sheaf of coinvariants will be parameterised by a choice of the fields  $S^{s_i}(z)$  for  $i = 1, \dots, \text{rank}(\mathfrak{g})$ .

**A correspondence between Hecke eigensheaves on  $\text{Bun}_G$  and flat  ${}^L G$ -bundles on  $\Sigma$ .** Finally, we shall now demonstrate that the above observation about a  $\mathfrak{z}(\widehat{\mathfrak{g}})$ -dependent realisation of the sheaf of coinvariants on  $\text{Bun}_G$ , and the isomorphism of Poisson algebras

$\mathfrak{z}(\widehat{\mathfrak{g}}) \simeq \mathcal{W}_\infty({}^L\widehat{\mathfrak{g}})$  discussed in section 4, will result in a correspondence between Hecke eigensheaves on  $\text{Bun}_G$  and flat holomorphic  ${}^L G$ -bundles on the worldsheet  $\Sigma$ .

Firstly, note that the classical  $\mathcal{W}$ -algebra  $\mathcal{W}_\infty({}^L\widehat{\mathfrak{g}})$  is isomorphic to  $\text{Fun Op}_{L\mathfrak{g}}(D^\times)$ , the algebra of functions on the space of  ${}^L\mathfrak{g}$ -opers on the punctured disc  $D^\times$  in  $\Sigma$ , where an  ${}^L\mathfrak{g}$ -oper on  $\Sigma$  is an  $n^{\text{th}}$  order differential operator acting from  $\Omega^{-(n-1)/2}$  to  $\Omega^{(n+1)/2}$  (where  $\Omega$  is the canonical line bundle on  $\Sigma$ ) whose principal symbol is equal to 1 and subprincipal symbol is equal to 0 [8]. Roughly speaking, it may be viewed as a (flat) connection on an  ${}^L G$ -bundle on  $\Sigma$ . In turn,  $\text{Fun Op}_{L\mathfrak{g}}(D^\times)$  is related to the algebra  $\text{Fun Op}_{L\mathfrak{g}}(D)$  of functions on the space of  ${}^L\mathfrak{g}$ -opers on the formal disc  $D$  in  $\Sigma$ , via  $\text{Fun Op}_{L\mathfrak{g}}(D^\times) \simeq \widetilde{U}(\text{Fun Op}_{L\mathfrak{g}}(D))$ , where  $\widetilde{U}$  is a functor from the category of vertex algebras to the category of Poisson algebras [33]. Since we have an isomorphism of Poisson algebras  $\mathfrak{z}(\widehat{\mathfrak{g}}) \simeq \mathcal{W}_\infty({}^L\widehat{\mathfrak{g}})$ , it will mean that the  $S^{(s_i)}(z)$ 's will correspond to the components of the (numeric)  ${}^L\mathfrak{g}$ -oper on the formal disc  $D$  in  $\Sigma$  [8]. Hence, a choice of the set of  $S^{(s_i)}(z)$  fields will amount to picking up an  ${}^L\mathfrak{g}$ -oper on  $D$ . Since any  ${}^L\mathfrak{g}$ -oper on  $D$  can be extended to a regular  ${}^L\mathfrak{g}$ -oper that is defined globally on  $\Sigma$ , it will mean that a choice of the set of  $S^{(s_i)}(z)$  fields will determine a unique  ${}^L G$ -bundle on  $\Sigma$  (that admits a structure of an oper  $\chi$ ) with a holomorphic connection.

Secondly, recall that we have a  $\mathfrak{z}(\widehat{\mathfrak{g}})$ -dependent realisation of the sheaf of coinvariants on  $\text{Bun}_G$  which depends on the choice of the fields  $S^{s_i}(z)$  for  $i = 1, \dots, \text{rank}(\mathfrak{g})$ . Hence, from the discussion in the previous paragraph, we see that we have a correspondence between a flat holomorphic  ${}^L G$ -bundle on  $\Sigma$  and a sheaf of coinvariants on  $\text{Bun}_G$ .

Lastly, recall that  $\Delta(V_\chi)$  — the sheaf of coinvariants on  $\text{Bun}_G$  — has a structure of a twisted  $\mathcal{D}$ -module on  $\text{Bun}_G$ . For a general group  $G$ , the sought-after Hecke eigensheaf [3] will be given by a  $\mathcal{D}$ -module  $\Delta(V_\chi) \otimes \Lambda_\chi^{-1}$  on  $\text{Bun}_G$  with eigenvalue  $E_\chi$ , where  $\Lambda_\chi$  is an invertible sheaf (i.e., a certain line bundle) on  $\text{Bun}_G$  equipped with a structure of a twisted  $\mathcal{D}$ -module, and  $E_\chi$  is the unique  ${}^L G$ -bundle corresponding to a particular choice of the set of  $S^{(s_i)}(z)$  fields. In the case where  $G$  is simply-connected, the Hecke eigensheaf will be given [8] by the *untwisted* holonomic  $\mathcal{D}$ -module  $\Delta(V_\chi) \otimes K^{-1/2}$  on  $\text{Bun}_G$  with eigenvalue  $E_\chi$ , where  $K$  is the canonical line bundle on  $\text{Bun}_G$ . In short, since tensoring with the invertible sheaf  $\Lambda_\chi$  or the canonical line bundle  $K$  on  $\text{Bun}_G$  just maps  $\Delta(V_\chi)$  to  $\Delta(V_\chi) \otimes \Lambda_\chi^{-1}$  or  $\Delta(V_\chi) \otimes K^{-1/2}$  in a one-to-one fashion respectively, we find that we have a one-to-one correspondence between a Hecke eigensheaf on  $\text{Bun}_G$  and a flat holomorphic  ${}^L G$ -bundle on  $\Sigma$ , where  $\Sigma$  is a closed Riemann surface of any genus, i.e., we have a geometric Langlands correspondence for  $G$ .<sup>11</sup>

**Physical interpretation of the Hecke eigensheaves on  $\text{Bun}_G$ .** From all of our

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<sup>11</sup>Note that the above-mentioned flat holomorphic  ${}^L G$ -bundles on  $\Sigma$  are restricted to those that have a structure of an  ${}^L\mathfrak{g}$ -oper on  $\Sigma$ . The space of connections of any such bundle only form a half-dimensional subspace in the moduli stack  $\text{Loc}_{L G}$  of the space of *all* connections on a particular flat  ${}^L G$ -bundle. Thus, our construction establishes the geometric Langlands correspondence only partially. However, it turns out that our construction can be generalised to include all flat  ${}^L G$ -bundles on  $\Sigma$  by considering in the correlation functions more general chiral operators that are labelled by finite-dimensional representations of  $\mathfrak{g}$ , which, in mathematical terms, is equivalent to making manifest the singular oper structure of any flat  ${}^L G$ -bundle on  $\Sigma$  [8].



above results, we see that one can physically interpret the Hecke eigensheaf as follows. A local section of the fibre of the Hecke eigensheaf over a point  $p$  in  $\text{Bun}_G$ , will determine, for some holomorphic  $G$ -bundle on  $\Sigma$  that corresponds to the point  $p$  in the moduli space  $\text{Bun}_G$  of all holomorphic  $G$ -bundles on  $\Sigma$ , the value of any  $n$ -point correlation function  $\langle \Phi_s^{\lambda_1}(z_1) \dots \Phi_s^{\lambda_n}(z_n) \rangle$  of local bosonic operators in the holomorphic chiral algebra of the twisted  $(0, 2)$  sigma-model on  $G/B$ . And the geometric Langlands correspondence for  $G$  just tells us that for every flat, holomorphic  ${}^L G$ -bundle that can be constructed over  $\Sigma$ , we have a unique way of characterising how an  $n$ -point correlation function of local bosonic primary operators in the holomorphic chiral algebra of a *quasi-topological* sigma-model with *no* boundaries like the twisted  $(0, 2)$  sigma-model on  $G/B$ , will vary under the *local*  $G$ -transformations generated by the affine  $J^a(z)$  currents on the worldsheet described earlier.

## 5.2 Hecke operators and the correlation functions of local operators

Consider the quantum operator  $\mathcal{J}(z) = \frac{1}{(-n_1-1)! \dots (-n_m-1)!} : \partial_z^{-n_1-1} J^{a_1}(z) \dots \partial_z^{-n_m-1} J^{a_m}(z) :$ . Note that since the  $J^a(z)$ 's are  $\overline{Q}_+$ -closed and in the  $\overline{Q}_+$ -cohomology or holomorphic chiral algebra of the sigma-model on  $G/B$ , so will  $\mathcal{J}(z)$  or polynomials  $F(\mathcal{J}(z))$  of arbitrary positive degree in  $\mathcal{J}(z)$  (modulo polynomials of arbitrary positive degree in the  $S^{(s_i)}(z)$  operators which necessarily act by zero and hence vanish in the quantum theory).<sup>12</sup>

The set of local operators described by  $F(\mathcal{J}(z))$  can be identified with the mathematically defined chiral vertex algebra  $V_{-h^\vee}(\mathfrak{g})$  associated to  $\widehat{\mathfrak{g}}$  at the critical level  $k = -h^\vee$ . The action of the Hecke operator on a Hecke eigensheaf as defined in the axiomatic CFT sense, is equivalent to an insertion of an operator that lies in the chiral vertex algebra given by  $m$  copies of  $V_{-h^\vee}(\mathfrak{g})$ , i.e.,  $\oplus_m V_{-h^\vee}(\mathfrak{g})$  [8]. Such an operator is again a polynomial operator of the form  $F(\mathcal{J}(z))$ . In short, the action of the Hecke operator is equivalent to inserting into the correlation functions of local primary field operators of the twisted  $(0, 2)$  sigma-model on  $G/B$ , other local operators that also lie in the holomorphic chiral algebra of the twisted  $(0, 2)$  sigma-model on  $G/B$ , which, as emphasised earlier, is a *quasi-topological* sigma-model with *no* boundaries. This is to be contrasted with the description of the Hecke operators (and Hecke eigensheaves) in the gauge-theoretic approach to the geometric Langlands program, where they are interpreted as 't Hooft line operators (and D-branes) in a *topological* sigma-model *with* boundaries. Our results therefore provide an alternative physical interpretation of these abstract objects of the geometric Langlands correspondence for  $G$ , to that furnished in the gauge-theoretic approach by Kapustin and Witten in [2].

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<sup>12</sup>In order to show this, first note that  $\partial_z J^a(z) = [L_{-1}, J^a(z)]$ , where  $L_{-1} = \oint dz T(z)$ . Since  $[\overline{Q}_+, J^a(z)] = 0$  even at the quantum level, it will mean that  $[\overline{Q}_+, \partial_z J^a(z)] = [[\overline{Q}_+, L_{-1}], J^a(z)] = \oint dz' [[\overline{Q}_+, T(z')], J^a(z)] = \oint dz' [\partial_{z'}(R_{i\bar{j}} \partial_{z'} \phi^i \psi^{\bar{j}}), J^a(z)] = 0$ . One can then repeat this argument and show that  $[\overline{Q}_+, \partial_z^m J^a(z)] = 0$  for any  $m \geq 1$  at the quantum level, always.

## 6. The cases with tame and mild ramifications

In this section, we shall discuss the cases of tame and mild ramifications in the geometric Langlands correspondence for  $G$ . We shall explain how, within our context, tamely-ramified, flat  ${}^L G$ -bundles on  $\Sigma$  will correspond to categories of Hecke eigensheaves on  $\text{Bun}_{G, \{y_1, \dots, y_k\}}$  — the moduli space of holomorphic  $G$ -bundles on  $\Sigma$  with parabolic structures at the points  $\{y_1, \dots, y_k\}$  in  $\Sigma$ . We will do this for mildly-ramified bundles as well. A physical interpretation of these Hecke eigensheaves in terms of the correlation functions of local operators in the holomorphic chiral algebra of the twisted sigma-model on  $X = G/B$ , will also be furnished.

### 6.1 Tamely-ramified ${}^L G$ -bundles on $\Sigma$ and the category of Hecke eigensheaves on $\text{Bun}_{G, y_1, \dots, y_k}$

In the case of tame ramification, the flat connection of the  ${}^L G$ -bundle over  $\Sigma$  will be modified. Specifically, at a set of points  $\{y_1, y_2, \dots, y_k\}$  on  $\Sigma$ , the connection will have regular singularities, i.e., it will contain a pole of order 1 at each point. In addition, as one traverses around each of these points, the connection will undergo a unipotent monodromy valued in the conjugacy class of  ${}^L G$ . For simplicity of argument, let us henceforth consider the case where we only have a single point  $y$ ; the story for multiple points will be analogous. One may then ask the following question: What does this tamely-ramified  ${}^L G$ -bundle on  $\Sigma$  correspond to in the context of the geometric Langlands correspondence for  $G$ ?

In order to answer this question, we will first need to revisit the unramified case. Recall that in the unramified case, the sheaf of coinvariants on  $\text{Bun}_G$  can be obtained purely mathematically as  $\Delta_x(V_{\chi_x})$ , where  $\Delta_x$  is a localisation functor, and where the subscript  $x$  is added for convenience to denote that  $D$  which appears in the relation  $\text{Fun Op}_{L\mathfrak{g}}(D^\times) \simeq \tilde{U}(\text{Fun Op}_{L\mathfrak{g}}(D))$ , is actually the formal disc at  $x \in \Sigma$ , such that  $\chi_x$  just reflects the restriction of the corresponding  ${}^L \mathfrak{g}$ -oper to  $D_x$ ; we omitted this specification earlier as our results in section 5 were independent of the point  $x$  — indeed, we have  $\Delta_x(V_{\chi_y}) \simeq \Delta_y(V_{\chi_x})$ , where  $y$  is any other point in  $\Sigma$  [8]. However, it will be useful to do so for our present discussion on tame ramification.

Note that the chiral vertex algebra  $V_{\chi_x}$  is formally called a  $(\widehat{\mathfrak{g}}_x, G_x)$ -module because it furnishes a representation of  $\widehat{\mathfrak{g}}_x$ , and because the centre  $\mathfrak{z}(\widehat{\mathfrak{g}}_x)$  commutes with the zero modes of  $\widehat{\mathfrak{g}}_x$  which generate the Lie algebra  $\mathfrak{g}_x$  of the group  $G_x$ . It can be viewed as an object in the category  $\mathcal{C}_{G_x, \chi_x}$  of  $(\widehat{\mathfrak{g}}_x, G_x)$ -modules. However, it follows from the results in [34] that  $\mathcal{C}_{G_x, \chi_x}$  is simply a category of vector spaces, and its unique up to isomorphism irreducible object is just  $V_{\chi_x}$ . As such, the localisation functor  $\Delta_x$  — which actually maps a category of objects to another category of objects — just maps  $V_{\chi_x}$  to a unique, irreducible Hecke eigensheaf on  $\text{Bun}_G$ , as discussed in section 5.

In the case where the  ${}^L G$ -bundle on  $\Sigma$  has a tame ramification at say the point  $y$ , the story will be somewhat different. The relevant oper which describes such a bundle is a *nilpotent*  ${}^L \mathfrak{g}$ -oper on  $D_y$  introduced in [35], and the space  $\text{Op}_{L\mathfrak{g}}^{\text{nil}}(D_y)$  of suchopers is a *subspace* of  $\text{Op}_{L\mathfrak{g}}(D_y)$ . Consequently, we have the relation  $\mathfrak{z}(\widehat{\mathfrak{g}}_y) \simeq \tilde{U}(\text{Fun Op}_{L\mathfrak{g}}^{\text{nil}}(D_y))$ , where  $\mathfrak{z}(\widehat{\mathfrak{g}}_y) \subset \mathfrak{z}(\widehat{\mathfrak{g}})$ .

In this ramified case, the object replacing  $V_{\chi_y}$  will be a  $(\widehat{\mathfrak{g}}_y, I_y)$ -module, where  $I_y$  is an Iwahori subgroup of the loop-group of  $G$  that is homomorphic to  $B$ , the Borel subgroup of  $G$  [36]; in axiomatic CFT language, the  $(\widehat{\mathfrak{g}}_y, I_y)$ -module is a Verma module of  $\widehat{\mathfrak{g}}_y$  at the critical level spanned by vectors which are  $I_y$ -invariant only. In contrast to the unramified case, the category  $\mathcal{C}_{I_y, \chi_y}$  of  $(\widehat{\mathfrak{g}}_y, I_y)$ -modules does not contain a unique irreducible object. Consequently, the localisation functor  $\Delta_y$  will map  $\mathcal{C}_{I_y, \chi_y}$  to a category  $\Delta_y(\mathcal{C}_{I_y, \chi_y})$  of Hecke eigensheaves.<sup>13</sup> A Hecke eigensheaf in this category will have an eigenvalue  $E_y$ , where  $E_y$  is a holomorphic  ${}^L G$ -bundle over  $\Sigma \setminus y$ .

One might now ask: on what kind of space is the above category of Hecke eigensheaves defined over? To answer this question, first note that the centre  $\mathfrak{z}(\widehat{\mathfrak{g}}_y)$  commutes with the Lie algebra  $\mathfrak{b}$  of  $B$  instead of the Lie algebra  $\mathfrak{g}$  of  $G$ . Since the centre  $\mathfrak{z}(\widehat{\mathfrak{g}}_y)$  is by definition what commutes with every element of  $\widehat{\mathfrak{g}}_y$ , it means that over the point  $y$ ,  $\widehat{\mathfrak{g}}_y$  is effectively  $\widehat{\mathfrak{b}}$ , the affine algebra of  $B \subset G$ ; this is consistent with  $\mathfrak{z}(\widehat{\mathfrak{g}}_y) \subset \mathfrak{z}(\widehat{\mathfrak{g}})$ . In other words, the commutator relation of (5.5) will reduce to the commutator relation for the Lie algebra  $\mathfrak{b}$ , at  $w = y$ . Via the exponential map discussed below (5.5), we see that we actually have a holomorphic  $G$ -bundle over  $\Sigma$  whose fibre at the point  $y$  will be reduced to  $B \subset G$  — that is, we have a holomorphic  $G$ -bundle on  $\Sigma$  with *parabolic structure* at the point  $y$  in  $\Sigma$ . Hence, the corresponding category of Hecke eigensheaves will be defined over  $\text{Bun}_{G,y}$  — the moduli space of holomorphic  $G$ -bundles on  $\Sigma$  with parabolic structure at  $y$ .

If we now consider another point  $x$  in  $\Sigma$  where there is no ramification of the  ${}^L G$ -bundle, the relevant category of modules will be given by  $\mathcal{C}_{G_x, \chi_x}$ . However, the category  $\Delta_x(\mathcal{C}_{G_x, \chi_x})$  cannot be supported over  $\text{Bun}_{G,y}$  — this is because  $\text{Bun}_{G,y}$  is an  $I_y$ -equivariant space, but  $\Delta_x(\mathcal{C}_{G_x, \chi_x})$  is not such a category. In other words, the category of *all* Hecke eigensheaves on  $\text{Bun}_{G,y}$  will be given by  $\Delta_y(\mathcal{C}_{I_y, \chi_y})$ .

Clearly, the above arguments can be easily extended to the multi-point case. In summary, in the geometric Langlands correspondence for  $G$  with tame ramification, we have a correspondence between a flat  ${}^L G$ -bundle that is tamely-ramified at a set of points  $\{y_1, \dots, y_k\}$  on  $\Sigma$ , and a category of Hecke eigensheaves on  $\text{Bun}_{G,y_1, \dots, y_k}$  — the moduli space of holomorphic  $G$ -bundles that have parabolic structures at the set of points  $\{y_1, \dots, y_k\}$  on  $\Sigma$ . In addition, a Hecke eigensheaf from the category will have an eigenvalue  $E_{y_1, \dots, y_k}$ , where  $E_{y_1, \dots, y_k}$  is a holomorphic  ${}^L G$ -bundle over  $\Sigma \setminus \{y_1, \dots, y_k\}$ .

## 6.2 Physical interpretation of Hecke eigensheaves on $\text{Bun}_{G,y_1, \dots, y_k}$

Recall from our discussion in section 5.1, that the variation of an arbitrary correlation function  $\langle \Phi_s^{\lambda_1}(z_1) \dots \Phi_s^{\lambda_n}(z_n) \rangle$  as one moves infinitesimal in  $\text{Bun}_G$ , will be given by  $\delta_\eta \langle \Phi_s^{\lambda_1}(z_1) \dots \Phi_s^{\lambda_n}(z_n) \rangle = \langle \oint_C dz \sum_a \eta_a(z) J^a(z) \Phi_s^{\lambda_1}(z_1) \dots \Phi_s^{\lambda_n}(z_n) \rangle$ . Also recall that this variation defines the manner in which the corresponding Hecke eigensheaf  $\Delta(V_\chi)^{14}$  will vary as one move along  $\text{Bun}_G$ , i.e., it defines a connection on the Hecke eigensheaf  $\Delta(V_\chi)$  over  $\text{Bun}_G$ .

<sup>13</sup>We have, for notational simplicity, omitted the factor  $\Lambda_{\chi_y}^{-1}$  that one is supposed to tensor with  $\Delta_y(\mathcal{C}_{I_y, \chi_y})$  to get a category of Hecke eigensheaves on the appropriate moduli space to be mentioned briefly.

<sup>14</sup>We have, for notational simplicity, omitted the factor  $\Lambda_\chi^{-1}$  that one is supposed to tensor with  $\Delta(V_\chi)$  to get a Hecke eigensheaf on  $\text{Bun}_G$ .

Certainly the connection on a Hecke eigensheaf over  $\text{Bun}_{G,y}$  will be different as the base space is no longer the same. Consequently, the variation of the correlation function  $\langle \Phi_s^{\lambda_1}(z_1) \dots \Phi_s^{\lambda_n}(z_n) \rangle$  has to be modified to express this difference. Essentially, one has to insert in the correlation function  $\langle \Phi_s^{\lambda_1}(z_1) \dots \Phi_s^{\lambda_n}(z_n) \rangle$ , a vertex operator which is associated — in the axiomatic CFT sense — to a highest weight vector in the  $(\widehat{\mathfrak{g}}_y, I_y)$ -module, at the point  $y$  in  $\Sigma$  [8].

Let us now ascertain what this vertex operator must correspond to in the context of the twisted sigma-model on  $X = G/B$ . Firstly, note that a  $(\widehat{\mathfrak{g}}_y, I_y)$ -module is given by a Verma module in the sense of axiomatic CFT [8]. Secondly, recall that the  $(\widehat{\mathfrak{g}}_y, I_y)$ -module consists only of  $I_y$ -invariant vectors. Thirdly, a highest weight vector  $\psi$  in a Verma module of an affine algebra  $\widehat{\mathfrak{g}}_y$ , is axiomatically defined as a state  $|\psi\rangle$ , where  $J_n^\alpha |\psi\rangle = 0$  for  $n > 0$ , and where the  $J_n^\alpha$ 's for  $\alpha = 1, \dots, \dim(\mathfrak{b})$  are the generators of  $\widehat{\mathfrak{g}}_y = \widehat{\mathfrak{b}}$  at  $y \in \Sigma$ . Altogether, this means that a vertex operator  $\varphi(z)$  of our interest, will be axiomatically represented by a state  $|\varphi\rangle$ , for which  $J_n^\alpha |\varphi\rangle = 0$  if  $n \geq 0$ . Notice that such a relation is realised by the OPE

$$J^\alpha(y) \cdot \varphi(w) \sim \text{regular}, \tag{6.1}$$

where  $y$  is fixed and  $w$  is variable in  $\Sigma$ . Notice that the regular term on the right-hand-side of (6.1) is a holomorphic function in  $w$ , and because  $\Sigma$  is a compact Riemann surface without boundaries, it will mean that this term is just a constant. Since a constant and the  $J^\alpha(y)$  currents are invariant under the symmetry transformations generated by the scalar supercharge  $\overline{Q}_+$  of the twisted sigma-model, (6.1) will imply that  $\varphi(w)$  is also  $\overline{Q}_+$ -invariant and in the  $\overline{Q}_+$ -cohomology. In fact,  $\varphi(w)$  corresponds to a class in  $H^0(X, \widehat{\mathcal{O}}_X^{\text{ch}})$ , i.e., it is a  $\psi^{\bar{j}}$ -independent operator in the  $\overline{Q}_+$ -cohomology — the cup product of sheaf cohomologies map products of global sections to global sections, and since the  $J^\alpha(y)$ 's correspond to global sections of  $\widehat{\mathcal{O}}_X^{\text{ch}}$ , and since for  $X = G/B$ , the space of dimension-zero global sections of  $\widehat{\mathcal{O}}_X^{\text{ch}}$  is one-dimensional and spanned by a constant [13], the OPE (6.1) will imply that  $\varphi(w)$  corresponds to a global section of  $\widehat{\mathcal{O}}_X^{\text{ch}}$ .

It is readily apparent that the above arguments can be easily extended to the multi-point case. In summary, the physical interpretation of a Hecke eigensheaf on  $\text{Bun}_{G,y_1,\dots,y_k}$  in the tamely-ramified case, will be as described in section 5 — that is, it is (up to a twist by the line bundle  $\Lambda_{\chi_y}^{-1}$ ) the sheaf of coinvariants spanned by vectors whose lengths-squared give us the values of the corresponding correlation functions of purely bosonic local operators  $\Phi_s^{\lambda_i}(z)$  in the holomorphic chiral algebra of the *closed, psuedo-topological* twisted sigma-model on  $X = G/B$  — the only difference being that one has to insert in the correlation functions the local operators  $\varphi(y_1), \varphi(y_2), \dots, \varphi(y_k)$  at the ramification points  $\{y_1, \dots, y_k\}$  in  $\Sigma$  which obey (6.1), where  $\varphi(y_1), \varphi(y_2), \dots, \varphi(y_k)$  are also in the holomorphic chiral algebra of the *closed, psuedo-topological* twisted sigma-model on  $X = G/B$ .

### 6.3 The case with mild ramification

Lastly, let us discuss the case with mild ramification. In this case, the flat connection of the  ${}^L G$ -bundle over  $\Sigma$  will instead have an irregular singularity at each of the points  $\{y_1, y_2, \dots, y_k\}$  on  $\Sigma$ , i.e., it will contain a pole of order  $p$ , where  $1 < p \leq n$  for some integer

$n$ , at each point. Again, for simplicity of illustration, let us consider the situation in which we only have a single point  $y$ ; the story for multiple points will be analogous.

In such a situation, one can just replace the Iwahori subgroup  $I_y$  with a congruence subgroup  $K_{m,y}$  (with  $m \geq n$ ) in the above arguments of section 6.1, and proceed as before [36]. Here,  $K_{m,y} = \exp(\mathfrak{g} \otimes (\mathfrak{m}_y)^m)$ , where  $\mathfrak{m}_y$  is the maximal ideal of the ring of integers  $\mathcal{O}_y$  at the point  $y$ .

In particular, this means that the corresponding category of Hecke eigensheaves will be defined over  $\widetilde{\text{Bun}}_{G,y}$ , the space of holomorphic  $G$ -bundles whose fibre is reduced to a subgroup of  $G$  that is homomorphic to  $K_{m,y}$  at the point  $y$  on  $\Sigma$ . In addition, a Hecke eigensheaf in this category has an eigenvalue  $\widetilde{E}$ , where  $\widetilde{E}$  corresponds to a flat  ${}^L G$ -bundle with mild ramification at the point  $y$  on  $\Sigma$ .

The physical interpretation of a Hecke eigensheaf in this mildly-ramified case will be somewhat similar as before; at the ramification point  $y$ , one will need to insert in the correlation function  $\langle \Phi_s^{\lambda_1}(z_1) \dots \Phi_s^{\lambda_n}(z_n) \rangle$ , a  $\psi^{\bar{j}}$ -independent local operator  $\tilde{\varphi}(y)$  in the holomorphic chiral algebra of the twisted sigma-model on  $X = G/B$ , that obeys

$$J^{\tilde{\alpha}}(y) \cdot \tilde{\varphi}(w) \sim \text{regular}, \tag{6.2}$$

where  $\tilde{\alpha} = 1, 2, \dots, \dim(\tilde{\mathfrak{g}})$ ;  $\tilde{\mathfrak{g}}$  being the Lie algebra of the subgroup of  $G$  that is homomorphic to  $K_{m,y}$ .

The case of mild ramification at multiple points in  $\Sigma$  is analogous as one can easily see via a straightforward extension of our above arguments. In summary, the physical interpretation of a Hecke eigensheaf on  $\widetilde{\text{Bun}}_{G,y_1,\dots,y_k}$  in the mildly-ramified case, will be as described in section 5 — that is, it is (up to a twist by the line bundle  $\Lambda_{X_y}^{-1}$ ) the sheaf of coinvariants spanned by vectors whose lengths-squared give us the values of the corresponding correlation functions of purely bosonic local operators  $\Phi_s^{\lambda_i}(z)$  in the holomorphic chiral algebra of the *closed, psuedo-topological* twisted sigma-model on  $X = G/B$  — the only difference being that one has to insert in the correlation functions the local operators  $\tilde{\varphi}(y_1), \tilde{\varphi}(y_2), \dots, \tilde{\varphi}(y_k)$  at the ramification points  $\{y_1, \dots, y_k\}$  in  $\Sigma$  which obey (6.2), where  $\tilde{\varphi}(y_1), \tilde{\varphi}(y_2), \dots, \tilde{\varphi}(y_k)$  are also in the holomorphic chiral algebra of the *closed, psuedo-topological* twisted sigma-model on  $X = G/B$ .

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